Conformal Field Theory with Boundaries and Defects

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1 Introduction

Symmetry plays a critical role in quantum field theory, and we often distinguish several different types. There are gauge symmetries – the $SU(3) \times SU(2) \times U(1)$ of the standard model for instance. There are global symmetries; consider the approximate SU(2) flavor symmetry of the up and down quarks. There are discrete symmetries, for example charge conjugation C, parity P, and time T reversal. Most important of all, perhaps, are the space-time symmetries of special relativity, also known as the Poincaré group. After all, relativistic quantum field theories were developed out of an intent to wed quantum mechanics and special relativity.

Given the prominence of the Poincaré group in relativistic quantum field theory, one is led to ask whether this group might in certain contexts be a subgroup of some larger group. The contexts in which the Poincaré group can be enlarged turn out to be surprisingly limited. There is in fact a theorem, proven in 1967 by Coleman and Mandula, that the Poincaré group can be combined with internal, continuous symmetries, such as the $SU(3)$ of the standard model, in only a trivial way, as a direct product. In other words, if one takes an element g from the Poincaré group and an element h from a continuous internal symmetry group, then $gh = hg$.

These lectures are about an important loop hole to the Coleman-Mandula Theorem: conformal symmetry. The proof of the theorem involves the scattering or S matrix, and if the theory contains only massless particles, for which the S matrix is a somewhat problematic concept, the Poincaré group can be enlarged to the conformal symmetry group. There are other loop holes to the Coleman-Mandula Theorem which we will not discuss here. The proof further assumes the symmetry is generated by a Lie algebra, while supersymmetry involves a generalization of a Lie algebra, called a Lie super-algebra. Discrete symmetries and spontaneously broken symmetries can both be used to extend the Poincaré group in nontrivial ways as well.

That the Poincaré group can be extended in a limited set of special ways suggests a special role for conformal symmetry. Indeed, it is important for critical phenomena in condensed matter and statistical physics. It also plays a central role in the renormalization group for quantum field theory. Last but not least, it is an essential technical tool in the development of string theory.

- *Critical Phenomena:* There are many statistical and condensed matter systems which undergo second order phase transitions. At the critical point, these systems often admit effective field theory descriptions which have conformal symmetry. One oft cited example is the Ising model in two dimensions, with spins $\sigma_i = \pm 1$ on sites of a square lattice. The nearest neighbor spins are allowed to interact, leading to a Hamiltonian

$$H = -J \sum_{\langle ij \rangle} \sigma_i \sigma_j \ . \tag{1.1}$$

  At high temperature, the spins are disordered. Their average value vanishes: $\langle \sigma \rangle = 0$. On the other hand, at low temperature the spins will pick an orientation $\langle \sigma \rangle \neq 0$. In fact, the two phases are related by Kramers-Wannier duality, and there is a second order phase transition between the high and low temperature phases at the self-dual point.

At this critical point, there are fluctuations at all scales, and the theory is invariant under changes of scale. These scale transformations are an important generator of the full conformal symmetry group as we will see later.

- *Renormalization Group:* Perhaps the most difficult aspect of quantum field theory (QFT) is that the rules depend on the energy scale. A famous example of this phenomena is the energy dependence of quantum chromodynamics (QCD). A theory of quarks and gluons, at high energies these quarks and gluons are nearly free particles. However, at low energy, the interaction strength grows and they condense to form mesons and baryons, for example the pions observed in cosmic rays or the protons and neutrons in the nucleus of the atom. This dependence on energy scale is called the renormalization group.

  In the limit of very high energy (UV) or low energy (IR), the QFT has a fixed point behavior, where it no longer depends on scale.[1] In the case of QCD, these fixed points have a simple nature. There is a free fixed point in the UV, where the particles cease to interact with each other, and a trivial fixed point in the IR. We say the IR fixed point is trivial because all of the bound states formed have mass. If we go to an energy scale below the mass of the lightest particle (a pion), there is not enough energy to produce any excitations, and the theory is empty, or trivial. Free and trivial are not the only options, however. It is possible to have a scale invariant, interacting theory of massless particles. These interacting conformal field theories are a major subject of these lectures and provide the generic fixed point behavior of a Lorentz invariant QFT. They are thus important starting points from which to begin the analysis of a general QFT.

- *String Theory:* The renormalization group is a way of curing the divergences that appear in generic QFT calculations. Intuitively, the problem is that point-like particles of relativistic QFTs are singular objects. The self energy of a charged point particle is infinite, and many other processes, for example scattering, generate similar infinities. Renormalization emerges from adding counter-terms to cure the divergences but that introduce a scale dependence to various physical quantities such as masses and coupling strengths. One might take the reasonable point of view that QFT, with its singular behavior and consequent scale dependence, is the wrong starting point for a fundamental description of the physical world. A theory of extended objects, for example strings, is somewhat less singular. Indeed string theory has emerged as one of the leading frameworks in which to unify the Standard Model of Particle Physics (open strings) with gravity (closed strings) at a quantum level. The string, as it propagates through time, traces out a 1+1 dimensional world sheet which hosts its own QFT. This QFT is a conformal field theory.

In these notes, we are interested further in breaking the conformal symmetry in a controlled but explicit way through the addition of boundaries and defects. If we start with a d dimensional space-time and add a $p < d$ dimensional defect, we will see that the residual symmetry is that of a p dimensional conformal field theory along with the transverse

[1]People have speculated about more general behavior, for example limit cycles, but such QFTs usually have additional pathologies. For example, they may be non-unitary.

rotation group around the defect. Such a breaking is very natural in the context of critical phenomena, where real world materials are necessarily finite in extent and further may well contain defects of various codimension. But there are additional more formal reasons for considering these extended objects. In hindsight, many of the most signficant developments in theoretical physics over the last thirty or forty years involve boundaries and defects in important ways. In the context of string theory, the discovery D-branes seeded the second superstring revolution. They gave us nonperturbative insight into the relations between the different super string theories, helping to show that through dualities, they were all part of a unified, larger structure. At the same time, D-branes by definition are boundary conditions for open strings. Thus they are boundaries for two dimensional conformal field theories.

AdS/CFT correspondence is another huge milestone in the development of theoretical physics, providing a map between gravity and quantum field theory, giving us insight both into quantum gravity and strongly interacting quantum field theories – two of the most important outstanding problems in theoretical physics today. At the same time, boundaries play a central role. The conformal boundary of anti de-Sitter (AdS) space in gravity is where the quantum field theory "lives". In a limit where we freeze the graviton and make the metric non-dynamical, we will see that this set-up is Weyl rescaling equivalent to the defect and boundary conformal field theories that play a central role in these notes.

Quantum entanglement in many body physics is another area where boundaries and more specifically defects play a central role. These ideas have become more prominent of late. Some hope to use quantum entanglement as a resource to build a quantum computer. Others hold out the hope that gravity and the geometry of space-time could emerge from information theoretic considerations, especially given our emerging understanding of black holes as thermodynamic objects that carry entropy and encode information. A common question is to ask about the entanglement between two spatial regions in a quantum field theory. To measure the entanglement, one often employs the replica trick, which is equivalent to inserting a codimension two defect into the space-time to implement an n-fold cover over the spatial regions in question.

Finally, topological insulators provide a last example where boundaries play a key role. In these experimentally realizable materials, an insulating bulk has conducting surface states. In more familiar QFT language, the quasiparticles in the bulk have mass (are "gapped") while the surface states are massless. Their massless nature is protected by discrete symmetries. More recently however, there has been hope of constructing related materials where the bulk also is gapless [[refs at beginning of Padayasi, Krishnan, Metlitski, Gruzberg, Meineri, 2111.03071]].

1.1 Dimensional Analysis

Dimensional analysis is a powerful tool in physics. It often allows you to deduce the answers to questions about which you have at best a foggy grasp of the details. A case in point is deducing the velocity of surface waves on a liquid – so-called capillary waves. These are the waves that you see moving away from a small stone that you toss in a lake, that travel maybe at a few dozen centimeters per second. Let's begin with the assumption that this speed should have something to do with the density of the liquid ρ, the surface tension σ, and the acceleration due to gravity g. If you further know that ρ is measured in mass per unit volume kg m^{-3}, σ in force per unit length kg s^{-2}, and acceleration in distance per unit of time squared m s^{-2}, then the unique quantity with units of velocity that can be constructed from these numbers is $(g\sigma/\rho)^{1/4}$. Plugging in the numbers for water, for which $\sigma = 72.8$ mN m^{-1}, one gets 16 cm s^{-1}, not bad for a back of the envelope estimate. Or one could turn this calculation around and estimate the surface tension for water from a stone throwing experiment at your local pond.

Similar dimensional analysis estimates will be crucial in our discussion of conformal and super symmetry in this class. We include a couple of problems to tone your skills.

Problem 1.1. *Using only the quantities $\hbar$, G_N, and c, construct quantities that have the units of length, mass, and time. Compute the corresponding Planck length, Planck mass, and Planck time, using SI units.*

Problem 1.2. *Another proposed source of extra physics is extra dimensions. Assume that we live not in a four dimensional world but a $(4+p)$-dimensional one where the extra dimensions are all extremely small circles of length ℓ.*

a) *Noting that the dimensionality of G_N is different in $(4+p)$ dimensions, what is the new expression for the Planck energy E_P in terms of $\hbar$, c, and G_N?*

b) *Find a relationship between G_N and the observed 4d value G_N^{4d}. Given the observed 4d value for G_N^{4d}, how small must ℓ be in order to have $E_P = 1$ TeV? Are there some values of p that you can rule out?*

For a relativistic quantum field theory, we almost always work in units where $\hbar$ and c are dimensionless quantities set equal to one. This choice gives time and distance the same units. It also gives momentum, energy, and mass the same units, and relates mass to one over distance, leaving us precisely one unit to work with, which we could either call length or mass.

To put these notions to work consider the action for a free scalar field:

$$S = -\int \mathrm{d}^d x \left((\partial_\mu \phi)(\partial^\mu \phi) + m^2 \phi^2 \right) \ . \tag{1.2}$$

From the knowledge that that action is dimensionless – after all e^{iS} must be a sensible expression in computing the path integral now that $\hbar = 1$ – we can conclude that ϕ has mass dimension

$$(\text{mass})^{\frac{d-2}{2}} \ .$$

We will often write this fact as $\Delta_\phi = \frac{d-2}{2}$, where Δ_ϕ is the scaling dimension of the field ϕ.

We can introduce an interaction to the theory by adding a $g\phi^n$ term to the Lagrangian. (Usually n is restricted to positive integer values to preserve analyticity.) Note that the coupling g will in general be dimensionful. To keep the interaction under control, we can try to keep it small and compute processes in a Taylor expansion in g. However, one should ask small compared to what? To address this question, we can make a dimensionless ratio $g/E^{d-n\Delta_\phi}$ where E is a characteristic energy of the process under consideration. The sign of $d - n\Delta_\phi$ then becomes of crucial importance. For $d - n\Delta_\phi > 0$, this dimensionless ratio becomes arbitrarily small at high energies but very large at low energies. Such an interaction is said to be *relevant* (i.e. relevant at low energies). In contrast, if $d - n\Delta_\phi < 0$, then the ratio becomes arbitrarily small at low energies but large at high energies. We say such an interaction is *irrelevant* (i.e. irrelevant at low energies). Note the mass m is *relevant* in this language, like in the case of QCD where the fact that all the particles have masses drives the theory to a gapped or trivial fixed point in the IR. (There is older nomenclature you may run into: *relevant* = *normalizable* and *irrelevant* = *non-normalizable*.)

The final case $d - n\Delta_\phi = 0$, called a classically marginal coupling, is important for our study of conformal symmetry and conformal field theory. In this case, g itself is dimensionless. Unfortunately, just because we can write such a term in a Lagrangian doesn't mean that g stays dimensionless at a quantum level. Typically loop corrections give anomalous dimensions to the quantum fields in a theory. And then $d - n\Delta_\phi$ is no longer zero. Nevertheless, these classically marginal theories very often provide tractable starting points for finding and analyzing CFTs. Their importance is conveyed by the fact that they are very often named after the people who first studied them in detail.

Problem 1.3. *Consider an interacting scalar field*

$$S = -\int \mathrm{d}^d x \left((\partial_\mu \phi)(\partial^\mu \phi) + g\phi^n \right) \ . \tag{1.3}$$

where n is a positive integer. For what pairs (n, d) can the coupling g be dimensionless?

Problem 1.4. *Consider the Lagrangian for a Dirac spinor in d dimensions*

$$S = -\int \mathrm{d}^d x \left(\frac{i}{2} \bar{\psi} \gamma^\mu \partial_\mu \psi + g(\bar{\psi}\psi)^n \right) \tag{1.4}$$

What is the scaling dimension of ψ? (You may assume the conjugate spinor $\bar{\psi}$ has the same scaling dimension as ψ. Moreover, the gamma matrices are dimensionless.) For what (n, d) can g be made dimensionless, assuming n is a positive integer? Considering now also the scalar field of the previous problem. In what dimensions do $\phi\bar{\psi}\psi$ and $\phi^2\bar{\psi}\psi$ lead to classically marginal couplings?

Problem 1.5. *Start with the assumption that the supersymmetry transformation Q squares to the momentum operator $Q^2 \sim P$ and moreover converts fermions into bosons and bosons into fermions. Try to guess how Q acts on ϕ and ψ, purely based on dimensional analysis.*

Problem 1.6. *Consider QED in d dimensions*

$$S = -\int \mathrm{d}^d x \left(\frac{1}{4} F^{\mu\nu} F_{\mu\nu} + \frac{i}{2} \bar{\psi} \gamma^\mu (\partial_\mu + igA_\mu) \psi \right) \tag{1.5}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. What is the scaling dimension of g? What is special about $d = 4$?

Problem 1.7. *Consider a scalar field that is free in the bulk but with interaction terms confined to a planar hypersurface M of dimension p:*

$$S = -\int \mathrm{d}^d x (\partial^\mu \phi)(\partial_\mu \phi) + \delta^{(d-p)}(x)\, g \int_M \mathrm{d}^p x \, \phi^n \ . \tag{1.6}$$

Identify some triples (d, p, n) for which g is classically marginal.

These kinds of finger counting exercises will be very valuable for us, in deciding whether or not a quantum field theory is conformally invariant, and in other situations as well. There is in fact an argument to be made that this subsection of the notes is the most important, with implications far beyond theoretical physics.

2 From Poincaré Symmetry to Conformal Symmetry

In this chapter, we will review the Poincaré group, the conformal group, and continuous internal symmetry groups, and then discuss how conformal symmetry evades the Coleman-Mandula theorem. (Space and time prevent us from including a proof of the theorem.)

The Poincaré group is a Lie group that is generated by space-time translations along with Lorentz transformations (which in turn consist of rotations and boosts). The infinitesimal version (or Lie algebra version) of this group action, under which the theory is invariant, can be written

$$x^\mu \to x^\mu + a^\mu + \omega^\mu{}_\nu x^\nu \ , \tag{2.1}$$

where the quantity $\delta x^\mu = a^\mu + \omega^\mu{}_\nu x^\nu$ is taken to be small.

In special relativity, the space-time proper distance $\Delta s^2 = \eta_{\mu\nu}\Delta x^\mu \Delta x^\nu$ between two points must be invariant under these transformations, which in turn places a constraint on $\omega^\mu{}_\nu$:

$$\begin{aligned}\Delta s^2 &\to \eta_{\mu\nu}(\Delta x^\mu + \omega^\mu{}_\lambda \Delta x^\lambda)(\Delta x^\nu + \omega^\nu{}_\rho \Delta x^\rho) \\ &= \eta_{\mu\nu}\Delta x^\mu \Delta x^\nu + \eta_{\mu\nu}\omega^\mu{}_\lambda \Delta x^\lambda \Delta x^\nu + \eta_{\mu\nu}\omega^\nu{}_\rho \Delta x^\mu \Delta x^\rho + \ldots \\ &= \Delta s^2 + (\omega_{\mu\nu} + \omega_{\nu\mu})\Delta x^\mu \Delta x^\nu + \ldots \ . \end{aligned} \tag{2.2}$$

In other words, $\omega_{\mu\nu} = -\omega_{\nu\mu}$ is antisymmetric under exchange of its indices.[2]

While elements of the Poincaré group compose to give new elements in the group, the infinitesimal version of this statement is that the commutator of two infinitesimal elements (i.e. elements of the corresponding Lie algebra) yields a new infinitesimal element. We consider infinitesimal elements δ_1 and δ_2 and compute

$$[\delta_1, \delta_2]x^\mu \equiv \delta_1 \delta_2 x^\mu - \delta_2 \delta_1 x^\mu \ . \tag{2.3}$$

To compute $\delta_2 \delta_1 x^\mu$, it is perhaps clearer to start with the arrow notation

$$\begin{aligned} x^\mu &\to x^\mu + a_1^\mu + \omega_{1\nu}^\mu x^\nu \\ &\to x^\mu + a_1^\mu + \omega_{1\nu}^\mu x^\nu + a_2^\mu + \omega_{2\nu}^\mu (x^\nu + a_1^\nu + \omega_{1\lambda}^\nu x^\lambda) \ , \end{aligned}$$

[2] We will use a Minkowski metric with mostly plus signature:

$$\eta_{\mu\nu} = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix} \ .$$

from which it follows that

$$\delta_2\delta_1 x^\mu = \omega^{\mu}_{2\,\nu} a_1^\nu + \omega^{\mu}_{2\,\lambda}\omega^{\lambda}_{1\,\nu} x^\nu + a_1^\mu + a_2^\mu + \omega^{\mu}_{1\nu} x^\nu + \omega^{\mu}_{2\nu} x^\nu \ . \tag{2.4}$$

Note the terms in red will drop out of the commutator. The commutator then must be

$$[\delta_1, \delta_2] x^\mu = (\omega^{\mu}_{1\,\lambda} a_2^\lambda - \omega^{\mu}_{2\,\lambda} a_1^\lambda) + (\omega^{\mu}_{1\,\lambda}\omega^{\lambda}_{2\,\nu} - \omega^{\mu}_{2\,\lambda}\omega^{\lambda}_{1\,\nu}) x^\nu \ . \tag{2.5}$$

The new infinitesimal Poincaré transformation is

$$a^\mu = \omega^{\mu}_{1\,\lambda} a_2^\lambda - \omega^{\mu}_{2\,\lambda} a_1^\lambda \ , \quad \omega^\mu{}_\nu = \omega^{\mu}_{1\,\lambda}\omega^{\lambda}_{2\,\nu} - \omega^{\mu}_{2\,\lambda}\omega^{\lambda}_{1\,\nu} \ . \tag{2.6}$$

Note that $\omega_{(\mu\nu)} = \frac{1}{2}(\omega_{\mu\nu} + \omega_{\nu\mu}) = 0$, consistent with the requirement that Δs^2 is invariant.

We would like to be able to act not just on space-time points x^μ with the Poincaré group but on quantum fields as well. To that end, we introduce the linear operators P_μ and $M_{\mu\nu}$ which act on the coordinates such that

$$\delta x^\mu = i a^\nu P_\nu(x^\mu) + \frac{i}{2}\omega^{\nu\lambda} M_{\nu\lambda}(x^\mu) \ . \tag{2.7}$$

The factor of 1/2 is introduced because of the anti-symmetry so that, for example, $\omega_{12} = -\omega_{21}$ is only counted once. The factors of i allow the generators to be Hermitian rather than anti-Hermitian operators. The commutator (2.5) can be written more abstractly as

$$\begin{aligned} [P_\mu, P_\nu] &= 0 \ , \\ [P_\mu, M_{\nu\lambda}] &= i\eta_{\mu\nu} P_\lambda - i\eta_{\mu\lambda} P_\nu \ , \\ [M_{\mu\nu}, M_{\lambda\rho}] &= i\eta_{\mu\lambda} M_{\nu\rho} - i\eta_{\nu\lambda} M_{\mu\rho} - i\eta_{\mu\rho} M_{\nu\lambda} + i\eta_{\nu\rho} M_{\mu\lambda} \ . \end{aligned} \tag{2.8}$$

Problem 2.1. *Reproduce the result (2.5) using P_μ and $M_{\nu\lambda}$ and in particular (2.7) and the commutator algebra (2.8).*

In general, we would like to be able to represent the action of P_μ and $M_{\mu\nu}$ not just on x^μ but on a quantum field $\Phi_I(x^\mu)$ which transforms under a representation of Poincaré and is additionally a function of a space-time point. Here I is some generalized index allowing for an arbitrary representation of the group. An infinitesimal group element of Poincaré g consisting of the data $(a_\mu, \omega_{\mu\nu})$ and acting on $\Phi_I(x^\mu)$ thus has two pieces, one g^{IJ} acting by matrix multiplication on the generalized index of the field I and the second acting on x^μ,

$$\delta\Phi_I(x^\mu) = g_I{}^J \Phi_J(x^\mu) + \Phi_I(x^\mu + \delta x^\mu) - \Phi_I(x^\mu) \ . \tag{2.9}$$

By a Taylor series, we can write the second two terms, to leading order, as a derivative

$$\Phi_I(x^\mu + \delta x^\mu) - \Phi_I(x^\mu) = (a^\mu + \omega^\mu{}_\nu x^\nu)\partial_\mu \Phi_I(x^\rho) \ . \tag{2.10}$$

Now it turns out that g^{IJ} simplifies as well and depends only on the Lorentz part of the Poincaré group. Because of the nontrivial commutator $[P_\mu, M_{\nu\lambda}]$, the Poincaré group is not a direct but a semi-direct product of translations and Lorentz transformations. Translations by themselves are straightforward to understand. They form an abelian and non-compact subgroup of the full group. Their irreducible representations are always one dimensional, and the corresponding matrices just constants. In fact, as far as I'm aware, for fields of physical interest, these constants always vanish. For example, for tensor fields, shifting the location of the origin of spacetime clearly should not affect the structure of the tangent and cotangent bundles, leaving the space-time indices on some general tensor field $T^{\mu_1\cdots\mu_n}_{\nu_1\cdots\nu_m}$ invariant.

The nontrivial data in g^{IJ} is then a representation of the Lorentz algebra only, and $P_\mu = -i\partial_\mu$ reduces to a derivative acting on the fields, controlling how the shift in x^μ in turn affects the field Φ_I. Smooth functions can be expanded in terms of a Taylor series:

$$\begin{aligned} f(x+a) &= f(x) + a^\mu \partial_\mu f(x) + \ldots \\ &= f(x) + i a^\mu P_\mu f(x) + \ldots \end{aligned} \tag{2.11}$$

Finite translations can be obtained as an exponential of P_μ:

$$\begin{aligned} f(x+a) &= e^{ia^\mu P_\mu} f(x) \\ &= f(x) + a^\mu \partial_\mu f(x) + \frac{1}{2} a^\mu a^\nu \partial_\mu \partial_\nu f(x) + \ldots \end{aligned} \tag{2.12}$$

The action of the Lorentz group on the coordinate dependence of Φ_I can be written in a similar derivative fashion, as $M_{\mu\nu} = i(x_\mu \partial_\nu - x_\nu \partial_\mu)$. Indeed, using this representation of $M_{\mu\nu}$ along with $P_\mu = -i\partial_\mu$, one can recover the commutation relations (2.8). However, this representation of the action of the Lorentz group on functions is not the whole story. The Lorentz group is non-abelian and admits more interesting representations. The Standard Model that we discussed briefly in the first section contains a Higgs field $H(x)$ in the trivial representation, vector fields such as the photon $A_\mu(x)$, and many spinor fields, such as the electron $\psi_\alpha(x)$. In general, a nontrivial representation of the Lorentz group implies that the field carries some kind of index, for example μ and α for the vector and spinor fields respectively. Different representations imply that there are different choices of matrices which satisfy the commutation relations (2.8) of the Poincaré group.

Problem 2.2. *For a vector representation, one takes*

$$(M_{\mu\nu})^\lambda{}_\rho = i\eta_{\mu\rho}\delta^\lambda_\nu - i\delta^\lambda_\mu \eta_{\nu\rho} \ . \tag{2.13}$$

(Notice that the indices μ and ν take a dual role, labeling both the Lorentz generator and its matrix components.) For the spinor representation, one takes instead

$$(M_{\mu\nu})_\alpha{}^\beta = -\frac{i}{2}(\gamma_{\mu\nu})_\alpha{}^\beta = -\frac{i}{4}(\gamma_\mu\gamma_\nu - \gamma_\nu\gamma_\mu)_\alpha{}^\beta \ , \tag{2.14}$$

where $(\gamma_\mu)_\alpha{}^\beta$ are the Dirac γ-matrices, $\{\gamma_\mu, \gamma_\nu\} = 2\eta_{\mu\nu}$. Verify that these two representations of the Lorentz group obey the commutation relations (2.8).

Quantum field theories often possess additional symmetries, most notably gauge symmetries. Associated with the gauged Lie group, there is a Lie algebra with commutation relations of the form

$$[T_a, T_b] = i f_{ab}{}^c T_c \ , \tag{2.15}$$

where the T_a are Hermitian generators, and ${f_{ab}}^c$ are the structure constants. The fields transform in representations of this algebra and carry associated indices. For example, the quarks ψ^a_α in the standard model in addition to a spinor index α carry an index a indicating that they transform in a fundamental representation of $SU(3)$.

The component P_t is both an energy and also a generator of infinitesimal translations in time. Because P_t exists as a well defined, time independent quantity, we expect that the total energy is conserved. Often a good first step in approaching a physics problem is to work out a complete set of conserved charges. In the context of our commutator algebra of P_μ, $M_{\mu\nu}$ and T_a, the set of conserved charges is the set which commutes with P_t. In the context of the Poincaré group, we expect the full four momentum P_μ to be conserved, along with angular momenta corresponding to M_{xy}, M_{yz}, and M_{zx}. The boosts M_{ti} on the other hand do not commute with P_t. Having written down the full set, as is typical in quantum mechanics one has to worry about whether the generators mutually commute as well. Otherwise, the operators will not all be simultaneously diagonalizable. In the context of spatial rotations, for example, one typically chooses $J_z = M_{xy}$ and the Casimir operator $J^2 = M_{xy}^2 + M_{yz}^2 + M_{zx}^2$.

From Noether's theorem, we expect that continuous symmetries are associated with conserved charges and more generally conserved currents. It should follow from Noether's theorem that $[P_t, T_a] = 0$. The content of the Coleman-Mandula theorem is much stronger, that the generators T_a commute with all of the generators of the Poincaré group:

$$[T_a, P_\mu] = 0 = [T_a, M_{\mu\nu}] \ . \tag{2.16}$$

Thus the T_a are not only conserved but transform under the trivial representation of the Poincaré group.

Theorem. (Coleman-Mandula) *In any spacetime dimension greater than two, the only interacting quantum field theories have Lie algebra symmetries which are a direct product of the Poincaré algebra with an internal symmetry.*

2.1 Conformal Symmetry

The proof of the Coleman-Mandula theorem relies on the existence of an S-matrix (or scattering matrix), which contains the data of all of the scattering amplitudes in the theory. A definition of the S-matrix requires the notion of asymptotic initial and final states, where the ingoing and outgoing particles are far from each other and essentially non-interacting. However, if the underlying theory is scale invariant, then there is no notion of "far", and there are difficulties in defining the S-matrix. One issue for the S-matrix is the presence of long range forces that occur when the particles that mediate those forces are massless. (Indeed, for a scale invariant theory, all the particles must be massless because a mass would define a scale.) You may have seen similar issues in a quantum mechanics class, in looking at the scattering cross section of a charged particle in a Coulomb potential. These problems stem from the masslessness of the photon. Scale invariant theories provide another important loop hole to the Coleman-Mandula theorem.

The Poincaré group was the set of transformations which left the Minkowski tensor $\eta_{\mu\nu}$ invariant. The conformal group is the set of coordinate transformations which leave the

Minkowski tensor invariant up to a position dependent rescaling

$$\eta'_{\mu\nu} \equiv \frac{\partial x^\alpha}{\partial x'^\mu}\frac{\partial x^\beta}{\partial x'^\nu}\eta_{\alpha\beta} = \Omega(x)\eta_{\mu\nu} \ . \tag{2.17}$$

Note the Poincaré group, for which $\Omega(x) = 1$ forms a subgroup of the conformal group. A further generator of the conformal group is the scale transformation $x^\mu \to x'^\mu = \lambda x^\mu$ for which $\Omega = \lambda^{-2}$. The rule (2.17) relates the Jacobian of the transformation to the scaling factor Ω, to wit $\Omega^d = J^2$. The word conformal is used to imply that the action of the group does not change the angle between intersecting curves. In the Euclidean context, when $\eta_{\mu\nu} = \delta_{\mu\nu}$, the cosine of the angle between two vectors is given by $v \cdot w/|v||w|$, and indeed, whether one is in the Euclidean or Minkowski signature, this quantity is invariant under conformal transformation.

Let us try to construct the infinitesimal elements of the conformal group. Consider a general coordinate transformation

$$x^\mu \to x'^\mu = x^\mu + \epsilon^\mu(x) \ , \tag{2.18}$$

assuming $\epsilon^\mu(x)$ is small. Using the rule

$$\eta'_{\mu\nu} = \frac{\partial x^\alpha}{\partial x'^\mu}\frac{\partial x^\beta}{\partial x'^\nu}\eta_{\alpha\beta} \ ,$$

we find to linear order that

$$\eta_{\mu\nu} \to \eta_{\mu\nu} - (\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu) + O(\epsilon^2) \ . \tag{2.19}$$

From the definition of a conformal transformation (2.17) with $\Omega(x) \approx 1 - f(x)$, we can make the identification

$$\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu = f(x)\eta_{\mu\nu} \ . \tag{2.20}$$

Taking a trace fixes

$$f(x) = \frac{2}{d}\partial_\rho \epsilon^\rho \ . \tag{2.21}$$

We would now like to establish what kinds of ϵ satisfy the constraint (2.20). To this end, we take a partial derivative ∂_ρ of (2.20) and permutations and construct the linear combination

$$\partial_\mu(\partial_\nu \epsilon_\rho + \partial_\rho \epsilon_\nu - f\eta_{\nu\rho}) + \partial_\nu(\partial_\mu \epsilon_\rho + \partial_\rho \epsilon_\mu - f\eta_{\mu\rho}) - \partial_\rho(\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu - f\eta_{\mu\nu}) = 0 \tag{2.22}$$

from which we conclude

$$2\partial_\mu \partial_\nu \epsilon_\rho = \eta_{\nu\rho}\partial_\mu f + \eta_{\mu\rho}\partial_\nu f - \eta_{\mu\nu}\partial_\rho f \ . \tag{2.23}$$

We further take a trace, which produces

$$2\partial^2 \epsilon_\rho = (2-d)\partial_\rho f \ , \tag{2.24}$$

indicating something rather special about conformal symmetry in two dimensions. We will specialize to the case $d > 2$ in the remainder of this argument.

We combine a symmetrized version of ∂_ν of (2.24)

$$\partial^2(\partial_\nu \epsilon_\mu + \partial_\mu \epsilon_\nu) = (2-d)\partial_\mu \partial_\nu f$$

along with ∂^2 of (2.20) to find

$$(2-d)\partial_\mu \partial_\nu f = \partial^2 f \eta_{\mu\nu} \ . \tag{2.25}$$

Finally taking a trace tells us that $(d-1)\partial^2 f = 0$ and hence that $\partial_\mu \partial_\nu f = 0$ vanishes, provided $d > 2$. In other words, f can be at most linear in the coordinates,

$$f = A + B_\mu x^\mu \ , \tag{2.26}$$

and ϵ_μ at most quadratic,

$$\epsilon_\mu = a_\mu + b_{\mu\nu} x^\nu + c_{\mu\nu\rho} x^\nu x^\rho \ , \tag{2.27}$$

with the restriction $c_{\mu\nu\rho} = c_{\mu\rho\nu}$. Plugging this ansatz into the constraint (2.20) yields the following conditions:

- a_μ is unconstrained and generates infinitesimal translations.
- $b_{\mu\nu} = \lambda \eta_{\mu\nu} + \omega_{\mu\nu}$ where $\omega_{\mu\nu} = -\omega_{\nu\mu}$ generate the Lorentz group and the trace part is an infinitesimal scale transformation.
- $c_{\mu\nu\rho} = \eta_{\mu\rho} b_\nu + \eta_{\mu\nu} b_\rho - \eta_{\nu\rho} b_\mu$ for a constant vector b_μ. These transformations are called special conformal transformations and act on coordinates as

$$x^\mu \to x'^\mu = x^\mu + 2(x\cdot b)x^\mu - b^\mu x^2 \ . \tag{2.28}$$

We give the finite versions of these infinitesimal transformations as table 2.1. Translations and Lorentz transformations generate the Poincaré group, as we have discussed at length. In total, we have d translations, $\frac{d(d-1)}{2}$ Lorentz transformations, one dilatation, and d generators of special conformal transformations for $\frac{(d+1)(d+2)}{2}$ generators in total. It is no accident that this number is the same as the dimension of the special orthogonal group $SO(d+2)$. There is an exercise a little later on to demonstrate that the conformal symmetry group is equivalent to $SO(d,2)$ (or $SO(d+1,1)$ in the Euclidean setting).

Problem 2.3. *Verify that $b_{\mu\nu} = \lambda \eta_{\mu\nu} + \omega_{\mu\nu}$ and $c_{\mu\nu\rho} = \eta_{\mu\rho} b_\nu + \eta_{\mu\nu} b_\rho - \eta_{\nu\rho} b_\mu$ are the only solutions for $b_{\mu\nu}$ and $c_{\mu\nu\rho}$ consistent with (2.20).*

Problem 2.4. *Verify that the infinitesimal versions of the transformations in table 2.1 recover a_μ, $b_{\mu\nu}$ and $c_{\mu\nu\rho}$.*

As the special coordinate transformations are somewhat ugly, it is often useful to introduce one further discrete element of the conformal group, the inversion

$$I : x^\mu \to x'^\mu = \frac{x^\mu}{x^2} \ , \tag{2.29}$$

where clearly I^2 is the identity element.

- translations: $x'^\mu = x^\mu + a^\mu$
- Lorentz: $x'^\mu = \Lambda^\mu{}_\nu x^\nu$
- dilatations (scale transformations): $x'^\mu = \lambda x^\mu$
- special conformal transformations: $x'^\mu = \frac{x^\mu - b^\mu x^2}{1 - 2b\cdot x + b^2 x^2}$

Figure 1: The finite versions of the generators of the conformal symmetry group.

Problem 2.5. *Demonstrate that an inversion followed by a translation followed by a further inversion is equivalent to a special coordinate transformation.*

Parallel to the earlier discussion of the Poincaré group, it is useful to have a more abstract presentation of the conformal group and its corresponding Lie algebra in terms of a set of generators and their commutation relations. Extending the Poincaré group to include dilatations D and special conformal transformations K_μ, we can write the transformation rule on a coordinate as

$$\delta x^\mu = i a^\nu P_\nu(x^\mu) + \frac{i}{2}\omega^{\nu\lambda} M_{\nu\lambda}(x^\mu) + i b^\nu K_\nu(x^\mu) + i\lambda D(x^\mu) \ . \tag{2.30}$$

From this expression, we infer how these transformations act on functions. We have $P_\mu = -i\partial_\mu$ and $M_{\mu\nu} = i(x_\mu \partial_\nu - x_\nu \partial_\mu)$ as we had before in the case of the Poincaré group, to which we add two more:

$$D = -i x^\mu \partial_\mu \ , \quad K_\mu = -i(2x_\mu x^\nu \partial_\nu - x^2 \partial_\mu) \ . \tag{2.31}$$

From this representation, it is then a straightforward although tedious exercise to work out how to extend the Poincaré group commutation relations to include the conformal group:

$$[D, P_\mu] = iP_\mu \ , \quad [D, K_\mu] = -iK_\mu \ , \quad [D, M_{\mu\nu}] = 0 \ , \quad [K_\mu, K_\nu] = 0 \ ,$$
$$[M_{\mu\nu}, K_\rho] = i(\eta_{\nu\rho} K_\mu - \eta_{\mu\rho} K_\nu) \ , \quad [P_\mu, K_\nu] = -2i(\eta_{\mu\nu} D + M_{\mu\nu}) \ .$$

Importantly, P_μ does not commute with dilatation or special conformal transformations, in apparent contradiction of the Coleman-Mandula theorem and also implying that massive states are not good eigenstates of the full conformal group.

One further remark is that $M_{\mu\nu}$, P_μ and D form a subgroup, and it is a subtle point whether there may exist theories which have scale invariance and Poincaré symmetry without also having the special conformal transformations. People have looked at this question in detail, and the lore seems to be that examples with scale but not conformal invariance are not physically interesting – they are non-unitary or have an unbounded spectrum or are non-interacting.

Problem 2.6. *Compute the commutator of P^2 with K_μ and D. What happens to a massive particle state $|p\rangle$ (where $P^2|p\rangle = m^2|p\rangle$, $m^2 \neq 0$) under the infinitesimal special conformal transformation K_μ?*

Problem 2.7. *If $\mu, \nu = 0, \dots, d-1$, then define $J_{\mu\nu} = M_{\mu\nu}$ along with $J_{\mu,d} = \frac{1}{2}(P_\mu - K_\mu)$, $J_{\mu,d+1} = \frac{1}{2}(P_\mu + K_\mu)$, and $J_{d,d+1} = D$, along with the constraint that $J_{ab} = -J_{ba}$ is antisymmetric. Show that the commutators of these generators are the same as for a $(d+2)$-dimensional orthogonal group, with metric signature $(2,d)$, i.e. $SO(2,d)$.*

Problem 2.8. *Write out the consistency relations (2.20) in $d = 2$ in the coordinate system $x_\pm = x \pm t$. What can you conclude about the allowed form of ϵ_μ?*

Problem 2.9. *Compute $\Omega(x)$ for the (finite) special conformal transformations.*

2.2 Boundaries and Defects

If we introduce a planar boundary and/or defect into the d-dimensional system, we will explicitly break the $SO(d,2)$ symmetry down to a subgroup. For simplicity in this section, let us work in Euclidean space, for which the signature of the conformal group shifts by one to $SO(d+1,1)$. For concreteness, introduce a p dimensional defect along the directions x^1, x^2, $\dots$, x^p at the location $x_{p+1} = 0$, $x_{p+2} = 0$, $\dots$, $x_d = 0$. We call $q = d - p$ the codimension of the defect. In codimension one, the difference between a boundary and a defect (or interface) is whether we restrict to $x_d > 0$ or not. We will often occasion to use the indices $a, b, c, \dots = 1, \dots p$ for tangential directions and $i, j, k, \dots = p+1, \dots p+q$ for normal directions.

Clearly the introduction of such a defect breaks the translation symmetries generated by P^μ, $\mu = p+1, \dots, d$. It will also break the special conformal transformations K^μ for the same range of indices $\mu = p+1, \dots, d$. The finite version of these transformations will move the defect around, either shifting its location or warping it into a sphere. The rotation group is broken into rotations within and around the defect, $SO(d) \to SO(p) \times SO(q)$. Dilatations are preserved. From this brief accounting, we see that the defect leads to a breaking of the conformal symmetry $SO(d+1,1) \to SO(p+1,1) \times SO(q)$. Operators that live on defects thus look like operators in a p-dimensional conformal field theory but that have an extra $SO(q)$ flavor symmetry that physically corresponds to their transverse spin.

In terms of generators, P_a, K_b, M_{ab}, M_{ij}, and D are preserved. P_i, K_j, and M_{ia} are broken. Note that it could be that some discrete remnant of M_{ia} could also be preserved, for example a rotation that flips the defect by 180 degrees. It's an interesting question. Whether or not such a discrete symmetry is there or not we leave up to the discretion of the reader and how they choose to define their theory.

The fact that the special conformal transformations change a planar defect into a spherical one means that spherical defects must preserve the same $SO(p+1,1) \times SO(q)$ subgroup of the full conformal group as the planar defects. It also suggests thinking about conformal field theories not just on flat space but also on other curved highly symmetric spaces, a topic we will have much to say about later on.

Problem 2.10. *Verify that a special conformal transformation that does not preserve the location of a defect will transform the defect into a spherical configuration.*

3 Constraints of Conformal Symmetry

We would like to understand how the conformal symmetry group acts on quantum states and fields. In the case of the Poincaré group, it is often convenient to choose fields that are eigenvectors of the momentum operator P_μ. In the context of the conformal symmetry group, P_μ no longer plays as privileged a role. P_μ does not commute with K_μ nor with D.

In the case of conformal symmetry, dilatation D largely replaces the privileged role of P^t. The commutation relations $[D, P_\mu] = iP_\mu$ and $[D, K_\mu] = -iK_\mu$ are suggestively close to the commutation relations for the raising and lower operators of the harmonic oscillator with the identifications $H \sim D$, $P_\mu \sim a^\dagger$ and $K_\mu \sim a$. Recall that for the harmonic oscillator, the raising and lower operators commute to give $[a, a^\dagger] = 1$ and the Hamiltonian can be written as a combination of these raising and lower operators: $H = a^\dagger a + E_0$, where E_0 is a constant (the ground state energy). A short computation leads to the conclusion $[H, a] = -a$ and $[H, a^\dagger] = a^\dagger$. If there is a lowest weight state $|0\rangle$, such that $a|0\rangle = 0$, then $H|0\rangle = E_0|0\rangle$. Moreover, the relation $H(a^\dagger)^n|0\rangle = (E_0 + n)(a^\dagger)^n|0\rangle$ follows from the commutation relations of H with $a^\dagger$.

We can play a very similar game with the conformal group. We declare a lowest weight state – or *primary* state – to be an eigenvector of the dilatation operator and also annihilated by special conformal transformations

$$
\begin{aligned}
D|\phi_I\rangle &= i\Delta|\phi_I\rangle \ , && (3.1)\\
K_\mu|\phi_I\rangle &= 0 \ . && (3.2)
\end{aligned}
$$

The factor of i is rather funny and is a consequence of the fact that the conformal group has indefinite signature, either $SO(d+1, 1)$ or $SO(d, 2)$, depending on whether we include a time-like direction. The dilatation operator does not have real eigenvalues!

If we like, we can also associate with the state an irreducible representation of the Lorentz group, indicated by the generalized index I. In this case, we have the rule that

$$
\frac{i}{2}\omega^{\mu\nu}M_{\mu\nu}|\phi_I\rangle = \frac{i}{2}\omega^{\mu\nu}(M_{\mu\nu})_J{}^I|\phi_I\rangle = g_I{}^J\phi_J\rangle \ , \tag{3.3}
$$

where by placing further I and J indices on $M_{\mu\nu}$, we have converted it from a generalized operator to a specific matrix representation of the Lorentz group. The rules for dealing with these more general representations of the Lorentz group quickly get involved, and so for the most part we will content ourselves with representations with no, one, or two vector indices.

Just as the harmonic oscillator has excited states that are formed by acting with $a^\dagger$ on the ground state, conformal primary states have *descendant* states which are constructed by acting with derivatives $P_\mu = -i\partial_\mu$ on the conformal primary state. Acting with P_μ n times increases the conformal weight $\Delta \to \Delta + n$. Acting with K_μ decreases the weight.

Most of the conformal field theory literature is phrased in terms of operators and correlation functions rather than states. We thus replace these conformal primary states with operators at the origin acting on the vacuum that create these states. A conformal primary operator $\phi_I(x)$ is one such that

$$
\phi_I(0)|0\rangle = |\phi_I\rangle \ . \tag{3.4}
$$

Part of the definition of the vacuum is that it is conformally invariant; it is annihilated by all of the generators of the conformal group. We could have chosen any point in space-time to insert the operator as all points are related via the conformal group. However, our choice of generators, for example $D = -ix^\mu \partial_\mu$, make the origin a simpler choice.

The action of the group on the operator is then given in terms of commutation relations:

$$[D, \phi_I(0)] = i\Delta \phi_I(0) \ , \tag{3.5}$$

$$[M_{\mu\nu}, \phi_I(0)] = (M_{\mu\nu})_I{}^J \phi_J(0) \ , \tag{3.6}$$

$$[K_\mu, \phi_I(0)] = 0 \ . \tag{3.7}$$

To recover the action of D, $M_{\mu\nu}$ and K_μ on $\phi_I(x)$ away from the origin, we use the fact that $\phi_I(x) = e^{iP\cdot x}\phi_I(0)e^{-iP\cdot x}$ and the commutator algebra of the conformal group. For instance

$$\begin{aligned} [D, \phi_I(x)] &= De^{iP\cdot x}\phi_I(0)e^{-iP\cdot x} - e^{iP\cdot x}\phi_I(0)e^{-iP\cdot x}D \\ &= e^{iPx}(e^{-iP\cdot x}De^{iP\cdot x}\phi_I(0) - \phi_I(0)e^{-iP\cdot x}De^{iP\cdot x})e^{-iP\cdot x} \\ &= e^{iP\cdot x}[\hat{D}, \phi_I(0)]e^{-iP\cdot x} \ , \end{aligned} \tag{3.8}$$

where we have defined $\hat{D} = e^{-iP\cdot x}De^{iP\cdot x}$. We then compute $\hat{D}$ explicitly,

$$\begin{aligned} \hat{D} &= \left(1 - ix\cdot P - \frac{(x\cdot P)^2}{2} + \ldots\right) D \left(1 + ix\cdot P - \frac{(x\cdot P)^2}{2} + \ldots\right) \\ &= D - ix^\mu[P_\mu, D] - \frac{1}{2}x^\mu x^\nu [P_\mu, [P_\nu, D]] + \ldots \end{aligned} \tag{3.9}$$

and from the commutator algebra conclude that $[P_\mu, [P_\mu, D]]$ and all higher order terms vanish. In short $\hat{D} = D - x^\mu P_\mu$ and

$$[D, \phi_I(x)] = i(\Delta + x^\mu \partial_\mu)\phi_I(x) \ . \tag{3.10}$$

A similar simplification occurs for the other elements of the conformal group.

Problem 3.1. *Verify that*

$$[K_\mu, \phi_I(x)] = -2ix_\mu \Delta \phi_I(x) - i(2x_\mu x^\nu \partial_\nu - x^2\partial_\mu)\phi_I(x) - 2x^\rho (M_{\rho\mu})_I{}^J \phi_J(x) \ . \tag{3.11}$$

From the infinitesimal action of the conformal group, one can in principle reconstruct the finite action on the field $\phi_I(x)$. For scalar fields (trivial representation of the Lorentz group), the rule is that

$$\phi'(x') = \Omega^{\Delta/2}\phi(x) \ . \tag{3.12}$$

Instead of constructing the finite version of the transformation from the infinitesimal one, it is more straightforward to check that the infinitesimal action of D and K_μ can be recovered from the finite transformations. Let us check D and leave K_μ for the reader. We want to look at the variation of the field at a particular point,

$$\delta\phi \equiv \phi'(x) - \phi(x) \ . \tag{3.13}$$

Note carefully which objects are primed and which are not in comparing this expression with (3.12). Now consider the dilatation $x' = (1+\lambda)x$ for small $\lambda \ll 1$. The infinitesimal change in the field is given by (3.10):

$$\delta\phi = i\lambda[D, \phi(x)] = -\lambda(\Delta + x^\mu \partial_\mu)\phi(x) \ . \tag{3.14}$$

For the dilatation $x' = (1+\lambda)x$, we know $\Omega = (1+\lambda)^{-2}$ and therefore from (3.12)

$$\phi'(x') = (1+\lambda)^{-\Delta}\phi(x) \approx (1 - \Delta\lambda)\phi(x) \ . \tag{3.15}$$

We could equally well consider the variation of the field at x' as at x:

$$\delta\phi(x') = \phi'(x') - \phi(x') \ . \tag{3.16}$$

We then expand out $\phi(x')$ in a Taylor series, $\phi(x') \approx \phi(x) + \lambda x^\mu \partial_\mu \phi(x)$, yielding

$$\delta\phi(x') \approx -\Delta\lambda\phi(x) - \lambda x^\mu \partial_\mu \phi(x) \ . \tag{3.17}$$

Then, because we are already working at linear order in λ, we are free to replace x on the right hand side with x', yielding the desired transformation rule.

Problem 3.2. *Verify that the rule (3.12) for the finite conformal symmetry transformations is also consistent with the infinitesimal transformation rule (3.11) for the special conformal transformations K_μ.*

For tensor fields, the power of Ω in the transformation rule is adjusted by the spin of the operator:

$$T'^{\nu_1\cdots\nu_n}_{\mu_1\cdots\mu_m}(x') = \Omega^{\frac{\Delta+n-m}{2}} \frac{\partial x'^{\nu_1}}{\partial x^{\beta_1}} \cdots \frac{\partial x'^{\nu_n}}{\partial x^{\beta_n}} \frac{\partial x^{\alpha_1}}{\partial x'^{\mu_1}} \cdots \frac{\partial x^{\alpha_m}}{\partial x'^{\nu_m}} T^{\beta_1\cdots\beta_n}_{\alpha_1\cdots\alpha_m}(x) \ . \tag{3.18}$$

3.1 A First Example and Noether's Theorem

Let us try to understand how the conformal symmetry group is realized for one of the simplest CFTs: a massless scalar field in d dimensions. We have the action for a real, massless scalar field

$$S = -\frac{1}{2}\int \mathrm{d}^d x \, (\partial^\mu\phi)(\partial_\mu\phi) \ .$$

Noether's Theorem guarantees that for every continuous symmetry of a quantum field theory, there exists (classically) a conserved current. One standard trick for identifying these conserved currents is to let the corresponding symmetry variation of the fields be a function of position: $\delta\phi(\lambda(x))$ which depends linearly on an infinitesimal parameter $\lambda(x)$. Because the infinitesimal transformation is a symmetry for constant λ, the variation of the action will take the form

$$\delta S = \int \mathrm{d}^d x (\partial_\mu \lambda) J^\mu \ .$$

If such a symmetry transformation acts only on the fields, then δS must be proportional to the equations of motion and must vanish, possibly after integration by parts, on-shell (i.e. after the application of the equations of motion). We conclude therefore that $\partial_\mu J^\mu = 0$, identifying the conserved current as J^μ. Let us see how this works in somewhat more detail for the conformal transformations acting on a free scalar.

Translations

For translations, we have

$$\delta\phi = \lambda(x) v^\mu \partial_\mu \phi(x) \ , \tag{3.19}$$

where v^μ is a constant vector of unit length. The action varies as

$$\begin{aligned} \delta S &= -\int \mathrm{d}^d x (\partial^\mu \phi) \partial_\mu (\lambda v^\nu \partial_\nu \phi) \\ &= \int \mathrm{d}^d x \, (\partial^\mu \lambda) v^\nu \left((\partial_\mu \phi)(\partial_\nu \phi) - \frac{1}{2} \eta_{\mu\nu} (\partial^\rho \phi)(\partial_\rho \phi) \right) \ . \end{aligned} \tag{3.20}$$

Thus we find

$$\tilde{J}_P^\mu = v_\nu \left((\partial^\mu \phi)(\partial^\nu \phi) - \frac{1}{2} \eta^{\mu\nu} (\partial^\rho \phi)(\partial_\rho \phi) \right) \ . \tag{3.21}$$

We use the subscript P to indicate a connection with the momentum generators. Note $\partial_\mu \tilde{J}_P^\mu = 0$ after applying $\Box \phi = 0$. Peeling off the factor of v_μ, we can identify the canonical stress-tensor

$$\tilde{T}_{\mu\nu} = (\partial_\mu \phi)(\partial_\nu \phi) - \frac{1}{2} \eta_{\mu\nu} (\partial^\rho \phi)(\partial_\rho \phi) \ . \tag{3.22}$$

(We leave the calculational details for $\tilde{J}_P^\mu$ as well as the other two currents $\tilde{J}_D^\mu$ and $\tilde{J}_K^\mu$ as an exercise below.)

Dilatations

For dilatations, noting $\Delta_\phi = \frac{d-2}{2}$, we have

$$\delta\phi = \lambda(x) \left(\Delta_\phi + x^\mu \partial_\mu \right) \phi(x) \ . \tag{3.23}$$

Here the variation of the action is slightly more involved:

$$\begin{aligned} \delta S &= -\int \mathrm{d}^d x (\partial^\mu \phi) \partial_\mu (\lambda (\Delta_\phi + x^\nu \partial_\nu) \phi) \\ &= -\int \mathrm{d}^d x (\partial_\mu \lambda) \left[(\partial^\mu \phi)(\Delta_\phi + x^\nu \partial_\nu) \phi - \frac{1}{2} \eta^{\mu\nu} x_\nu (\partial^\rho \phi)(\partial_\rho \phi) \right] \ . \end{aligned} \tag{3.24}$$

So we find

$$\tilde{J}_D^\mu = -\Delta_\phi (\partial^\mu \phi) \phi - x^\nu (\partial^\mu \phi)(\partial_\nu \phi) + \frac{1}{2} x^\mu (\partial^\rho \phi)(\partial_\rho \phi) \ . \tag{3.25}$$

Using $\Box \phi = 0$, the current is clearly conserved, $\partial_\mu \tilde{J}_D^\mu = 0$.

Intriguingly, we can identify $\tilde{J}_D^\mu = -x_\nu \tilde{T}^{\mu\nu} - \frac{d-2}{4} \partial^\mu \phi^2$. There is a usual story here that these currents are only well defined up to the divergence of an anti-symmetric two tensor, $\partial_\nu f^{\nu\mu}$ where $f^{\mu\nu} = -f^{\nu\mu}$. Shifting $\tilde{J}^\mu \to \ J^\mu = \tilde{J}^\mu + \partial_\nu f^{\nu\mu}$ does not interfere with conservation because $\partial_\mu \partial_\nu f^{\mu\nu} = 0$. In the present case, one often considers the following improvement of the dilatation current, $f_{\nu\mu} = \frac{d-2}{4(d-1)}(x_\nu \partial_\mu \phi^2 - x_\mu \partial_\nu \phi^2)$, leading to

$$J_D^\mu = -x_\nu \left(\tilde{T}^{\mu\nu} - \frac{d-2}{4(d-1)} (\partial^\mu \partial^\nu - \eta^{\mu\nu} \partial^2) \phi^2 \right) \tag{3.26}$$

The quantity in parentheses is the "improved stress tensor":

$$T^{\mu\nu} = \tilde{T}^{\mu\nu} - \frac{d-2}{4(d-1)} (\partial^\mu \partial^\nu - \eta^{\mu\nu} \partial^2) \ , \tag{3.27}$$

improved in the sense that it is traceless $T^\mu_\mu = 0$ on-shell. It is still conserved. In the form $J^\mu = x_\nu T^{\nu\mu}$, conservation is easier to verify as it follows diretly from conservation $\partial_\mu T^{\mu\nu} = 0$ and tracelessness $T^\mu_\mu = 0$ of the improved stress tensor.

Special Conformal Transformations

For special conformal transformations, we consider the variation

$$\delta\phi = \lambda(x) v^\mu \left(2\Delta_\phi x_\mu + 2x_\mu x^\nu \partial_\nu - x^2 \partial_\mu \right) \phi \ . \tag{3.28}$$

Here the variation of the action

$$\delta S \ = \ -\int \mathrm{d}^d x (\partial^\mu \phi) \partial_\mu (\lambda v^\rho \left(2\Delta_\phi x_\rho + 2x_\rho x^\nu \partial_\nu - x^2 \partial_\rho \right) \phi) \ , \tag{3.29}$$

leads us to the identification of the conserved current

$$\tilde{J}_K^\mu \ = \ -v^\rho (2x_\rho x_\nu - \eta_{\rho\nu} x^2) \tilde{T}^{\nu\mu} - (\partial^\mu \phi) \Delta_\phi (2v \cdot x) \phi + v^\mu \Delta_\phi \phi^2 \ .$$

With a little effort, one can explicitly verify that $\partial_\mu J_K^\mu = 0$. There is a slightly more complicated improvement term

$$f^{\nu\mu} = \frac{\Delta_\phi}{d-1} (2v \cdot x x^\nu \partial^\mu - x^2 v^\nu \partial^\mu + v^\nu x^\mu) \phi^2 - (\mu \leftrightarrow \nu) \tag{3.30}$$

which allows one to write the current in the improved form

$$J_K^\mu = -v^\rho (2x_\rho x_\nu - \eta_{\rho\nu} x^2) T^{\nu\mu} \ . \tag{3.31}$$

Problem 3.3. *For the massless scalar field, fill in the details above of the computations of the conserved currents J_P^μ, J_D^μ and J_K^μ that follow from the translations, dilatations and special conformal transformations. Explicitly verify conservation $\partial_\mu J^\mu = 0$ in each case.*

Integrating the charge density J^0 over a spatial slice S yields a conserved charge Q. We have the usual argument that Q is time independent:

$$\frac{\partial}{\partial t} Q = \frac{\partial}{\partial t} \int_S d^{d-1} x \, J^0 = -\int_S d^{d-1} x \, \partial_i J^i = 0 \tag{3.32}$$

by Stokes' Theorem, assuming the currents J^i fall off to zero at the boundary of S. In the current context, we identify the charges associated with J^μ_P, J^μ_D and J^μ_K as the generators of the conformal transformations P^μ, D, and K^μ respectively. Indeed, starting with the equal time canonical commutation relation for ϕ and its conjugate momentum operator $\partial_t \phi$, an industrious student can verify the commutation relations of the generators P^μ, D, and K^μ. A complete treatment will require adding also the rotation generators $M^{\mu\nu}$ that we did not consider here.

Two other simple examples of conformal field theories are a massless fermion ψ_α in d dimensions and a Maxwell field $F_{\mu\nu}$ in four dimensions. Another good exercise for the industrious student is to compute the corresponding conserved currents and charges for these theories. More ambitiously, starting from the canoncial commutation relations for the fields, one can try to verify the commutations relations of the conformal generators.

Free Fields with a Boundary

By restricting the d-dimensional scalar field to the region $x^{d-1} > 0$, we introduce a boundary at $x^{d-1} = 0$. The normal-tangential components of the improved stress tensor have the form

$$T^{na} = (\partial^n \phi)(\partial^a \phi) - \frac{d-2}{4(d-1)} \partial^n \partial^a \phi^2 \ , \tag{3.33}$$

where we denote the tangential indices $a, b = 0, 1, \ldots, d-2$ and the normal index by $x^n = x^{d-1}$. In order for the P^a and K^a generators to be conserved charges, it had better be that T^{na} vanishes at the boundary $x^n = 0$.[3] There are two obvious ways to guarantee this vanishing: Dirichlet $\phi = 0$ and Neumann $\partial_n \phi = 0$ boundary conditions. Indeed, these two boundary conditions compatible with the residual conformal $SO(d-1, 2)$ symmetry remaining after the introduction of a boundary.

We could also consider a Maxwell field in 4d, which has the action

$$S = -\frac{1}{4} \int_{x^n > 0} \mathrm{d}^4 x \, F^{\mu\nu} F_{\mu\nu} \ , \tag{3.34}$$

and stress tensor $T^{\mu\nu} = F_\mu{}^\lambda F_{\nu\lambda} - \frac{1}{4} \eta_{\mu\nu} F^{\lambda\rho} F_{\lambda\rho}$. Now the normal-tangential components are $T^{na} = F_n{}^\lambda F_{a\lambda}$, suggesting that the conformal boundary conditions are either "Dirichlet" $F_{ab} = 0$ or "Neumann" $F_{na} = 0$.

3.2 Correlation Functions

In the study of quantum field theory, a central role is played by the notion of a correlation function. These correlation functions are defined through the path integral

$$\langle \phi(x_1) \phi(x_2) \cdots \phi(x_n) \rangle \equiv \frac{1}{Z} \int [d\phi] \phi(x_1) \cdots \phi(x_n) e^{iS[\phi]} \ , \tag{3.35}$$

[3]Although this statement is clear enough in the present context of a free field theory with no additional degrees of freedom on the boundary, it is a bit subtle to extend it in full generality to situations where there may be boundary degrees of freedom that carry momentum and energy. We will be better equipped to deal with these subtleties later after discussing CFTs in curved space and something called the displacement operator.

for a generic action $S[\phi]$ that is a functional of a field $\phi(x)$. Here $Z = \int [d\phi] e^{iS[\phi]}$. We are interested in QFTs that are invariant with respect to a symmetry. That means, at a quantum level, both S and the measure $[d\phi]$ should be invariant with respect to the action of the symmetry group. (Theories where S is invariant but the measure fails to be invariant are said to have the symmetry classically but possess an anomaly.) These symmetries have consequences for the correlation functions, consequences which are called Ward identities.

Let us suppose that the symmetry acts on ϕ via $\phi \to R(\phi)$. We would like to understand how the symmetry affects the correlation function:

$$\langle R(\phi(x_1)) R(\phi(x_2)) \cdots R(\phi(x_n)) \rangle = \frac{1}{Z} \int [d\phi] R(\phi(x_1)) \cdots R(\phi(x_n)) e^{iS[\phi]} \ . \tag{3.36}$$

With respect to earlier notation $R(\phi(x)) = \phi'(x')$. Because the measure and the action are invariant under the symmetry, we can make the replacements $[d\phi] = [dR(\phi)]$ and $S[\phi] = S[R(\phi)]$ without changing the value of the correlation function:

$$\langle R(\phi(x_1)) R(\phi(x_2)) \cdots R(\phi(x_n)) \rangle = \frac{1}{Z} \int [dR(\phi)] R(\phi(x_1)) \cdots R(\phi(x_n)) e^{iS[R(\phi)]} \ .$$

Further, it is important to realize that $R[\phi]$ is just a dummy integration variable. We are free to replace it with ϕ itself. However, for space-time symmetries, this replacement will not affect the action of the symmetry group on the locations x_i of the $\phi(x_i)$ insertions:

$$\langle R(\phi(x_1)) R(\phi(x_2)) \cdots R(\phi(x_n)) \rangle = \frac{1}{Z} \int [d\phi] \phi(R(x_1)) \cdots \phi(R(x_n)) e^{iS[\phi]} \ .$$

We are left with the result, slightly generalizing to the case where the fields are distinct,

$$\langle \phi_1(R(x_1)) \phi_2(R(x_2)) \cdots \phi_n(R(x_n)) \rangle = \langle R(\phi_1(x_1)) R(\phi_2(x_2)) \cdots R(\phi_n(x_n)) \rangle \ . \tag{3.37}$$

For conformal symmetry and scalar primary operators, we can put (3.37) and (3.12) together to learn that

$$\langle \phi_1(x_1') \cdots \phi_n(x_n') \rangle = \left(\prod_{i=1}^{n} \Omega^{\Delta_i/2}(x_i) \right) \langle \phi_1(x_1) \cdots \phi_n(x_n) \rangle \ . \tag{3.38}$$

In the case of translations and Lorentz transformations, we have that $\Omega = 1$. For translations more particularly, we find that the correlation function depends only on the relative positions of the insertions

$$\langle \phi(x_1 + a) \cdots \phi(x_n + a) \rangle = \langle \phi(x_1) \cdots \phi(x_n) \rangle \ . \tag{3.39}$$

For Lorentz transformations, we are free to perform a global rotation and/or boost on the insertion points without affecting the answer:

$$\langle \phi(\Lambda \cdot x_1) \cdots \phi(\Lambda \cdot x_n) \rangle = \langle \phi(x_1) \cdots \phi(x_n) \rangle \ . \tag{3.40}$$

Assuming that the correlation function will depend on the x_i^μ, we must consider them pair-wise $x_i^\mu - x_j^\mu$ to remove the dependence on the translation parameter a^μ, and the indices must all be contracted in a Lorentz invariant way to avoid dependence on $\Lambda^\mu{}_\nu$.

But we have two more transformations at our disposal – dilatations and special conformal transformations – which turn out to be strong enough to fix the form of two and three point functions of scalar primaries up to constants. Let us see how these constraints arise in more detail.

Two Point Functions

For two point functions of scalars, Poincaré invariance implies the correlation function can only depend on the Lorentz invariant distance between the insertions

$$\langle \phi_1(x_1)\phi_2(x_2)\rangle = f(|x_1 - x_2|) \ . \tag{3.41}$$

Scale transformations $x \to x' = \lambda x$ further imply

$$\langle \phi_1(x_1)\phi_2(x_2)\rangle = \lambda^{\Delta_1+\Delta_2}\langle \phi_1(\lambda x_1)\phi_2(\lambda x_2)\rangle \ , \tag{3.42}$$

from which we conclude $f(|x_1 - x_2|) = \lambda^{\Delta_1+\Delta_2} f(\lambda|x_1 - x_2|)$. The only way to satisfy this constraint is to choose

$$\langle \phi_1(x_1)\phi_2(x_2)\rangle = \frac{C_{12}}{|x_1 - x_2|^{\Delta_1+\Delta_2}} \ . \tag{3.43}$$

Finally, we consider special conformal transformations. From Problem 2.9, you should have learned that

$$\Omega = (1 - 2b\cdot x + b^2x^2)^2 \tag{3.44}$$

Let us define $\gamma_i \equiv 1 - 2b\cdot x_i + b^2x_i^2$. A remarkable property about special conformal transformations is that

$$|x_1' - x_2'| = \frac{|x_1 - x_2|}{\gamma_1^{1/2}\gamma_2^{1/2}} \ , \tag{3.45}$$

from which we can see that

$$\frac{C_{12}}{|x_1 - x_2|^{\Delta_1+\Delta_2}} = \frac{1}{\gamma_1^{\Delta_1}\gamma_2^{\Delta_2}}\frac{C_{12}}{|x_1' - x_2'|^{\Delta_1+\Delta_2}} = \frac{(\gamma_1\gamma_2)^{\frac{\Delta_1+\Delta_2}{2}}}{\gamma_1^{\Delta_1}\gamma_2^{\Delta_2}}\frac{C_{12}}{|x_1 - x_2|^{\Delta_1+\Delta_2}} \tag{3.46}$$

where in the first equality, we used the Ward identity. This expression can only make sense if $\Delta_1 = \Delta_2$ or if $C_{12} = 0$, since γ_1 and γ_2 are independent quantities. The final result for the correlation function of two scalar primary operators is thus

$$\langle \phi_1(x_1)\phi_2(x_2)\rangle = \begin{cases} 0 & \Delta_1 \neq \Delta_2 \\ \frac{C_{12}}{|x_1 - x_2|^{2\Delta_1}} & \Delta_1 = \Delta_2 \ . \end{cases} \tag{3.47}$$

Often it is possible to normalize the fields such that $C_{12} = 1$. For example, for the free scalar field $\phi(x)$, a kinetic term in the action normalized with a $1/2$ in front will lead to a particular value of $C_{\phi\phi}$. However, by sending $\phi \to \phi' = c\phi$, one will shift the normalization $C_{\phi\phi} \to C_{\phi\phi}/c^2$.

Problem 3.4. *Verify the remarkable property (3.45).*

Three Point Functions

Three point functions $\langle\phi_1(x_1)\phi_2(x_2)\phi_3(x_3)\rangle$ of scalar primary operators are fixed in a similar manner. Poincaré plus scale invariance fix the correlation function to be a sum over terms of the form

$$\frac{1}{|x_{12}|^a|x_{23}|^b|x_{13}|^c} \ , \tag{3.48}$$

where $a+b+c = \Delta_1+\Delta_2+\Delta_3$. We have also introduced the compact notation $x_{ij} = |x_i - x_j|$. Special conformal invariance then fixes one particular choice of the constants a, b, and c. In particular, one finds the constraint

$$\frac{C_{123}}{|x_{12}|^a|x_{23}|^b|x_{13}|^c} = \frac{(\gamma_1\gamma_2)^{a/2}(\gamma_2\gamma_3)^{b/2}(\gamma_3\gamma_1)^{c/2}}{\gamma_1^{\Delta_1}\gamma_2^{\Delta_2}\gamma_3^{\Delta_3}}\frac{C_{123}}{|x_{12}|^a|x_{23}|^b|x_{13}|^c} \tag{3.49}$$

For this ratio of gamma factors to be unity,

$$\begin{aligned} a &= \Delta_1+\Delta_2-\Delta_3 \ , \\ b &= \Delta_2+\Delta_3-\Delta_1 \ , && (3.50) \\ c &= \Delta_3+\Delta_1-\Delta_2 \ . && (3.51) \end{aligned}$$

The final result is that

$$\langle\phi_1(x_1)\phi_2(x_2)\phi_3(x_3)\rangle = \frac{C_{123}}{|x_{12}|^{\Delta_1+\Delta_2-\Delta_3}|x_{23}|^{\Delta_2+\Delta_3-\Delta_1}|x_{31}|^{\Delta_3+\Delta_1-\Delta_2}} \ . \tag{3.52}$$

While the coefficients of two-point functions can often be absorbed through changing the normalization of the fields, the ratios of three point function coefficients C_{ijk} to two-point coefficients C_{ij} contain physical information.

Four Point Functions

Once we have four positions at our disposal, something new occurs. We can form the invariant cross ratios

$$u = \frac{x_{12}^2x_{34}^2}{x_{13}^2x_{24}^2} \ , \quad v = \frac{x_{14}^2x_{23}^2}{x_{13}^2x_{24}^2} \ , \tag{3.53}$$

which are invariant under the full conformal group. Unlike the two and three point functions, the four point function is not completely fixed by conformal invariance

$$\langle\phi_1(x_1)\phi_2(x_2)\phi_3(x_3)\phi_4(x_4)\rangle = \frac{F(u,v)}{\prod_{i<j}|x_{ij}^2|^{\delta_{ij}}} \ , \tag{3.54}$$

where $\sum_{j\neq i}\delta_{ij} = \Delta_i$. The function $F(u,v)$ is not constrained in any obvious way from this point of view.

Vector and Tensor Operators

One can play the same game with operators in nontrivial representations of the Lorentz group. Two important example worth mentioning are a conserved current J^μ and the stress tensor $T^{\mu\nu}$. Conservation here implies that $\partial_\mu J^\mu = 0$ and $\partial_\mu T^{\mu\nu} = 0$, which places further constraints on the correlation functions.

Let us begin with $\langle J_\mu(x) J_\nu(0)\rangle$, where by translation invariance, we are free to put the second current at the origin without loss of generality. The game is played by trying to construct the most general symmetric two index tensor out of the elementary building blocks available to us, in this case $\eta_{\mu\nu}$ and x^μ. Poincaré and scaling symmetry tell us that the two-point function must have the form

$$\langle J^\mu(x) J^\nu(0)\rangle = \tau \frac{\eta^{\mu\nu} + \alpha \frac{x^\mu x^\nu}{x^2}}{|x|^{2\Delta}} \tag{3.55}$$

where τ and α are constants. The general transformation rule for a vector field is

$$J'^\mu(x') = \frac{\partial x'^\mu}{\partial x^\nu} \Omega^{\frac{\Delta+1}{2}} J^\nu(x) \ . \tag{3.56}$$

Combined with the Ward identity $\langle J^\mu(x') J^\nu(y')\rangle = \langle J'^\mu(x') J'^\nu(y')\rangle$ for special conformal transformations, we find $\alpha = -2$. The tensor

$$I^{\mu\nu}(x) = \eta^{\mu\nu} - 2\frac{x^\mu x^\nu}{x^2} \ , \tag{3.57}$$

called the inversion tensor, plays an important role in conformal field theory.

Finally, we enforce the conservation condition $\partial_\mu \langle J^\mu(x) J^\nu(0)\rangle = 0$, which tells us either $\tau = 0$ or $\Delta = d-1$. In other words, conserved currents must have scaling dimension $d-1$, which makes sense from a dimensional analysis point of view. The time component J^0 is a charge density, which carries some units of dimensionless charge per unit volume.

The stress tensor two-point function can also be expressed in terms of the inversion tensor. One finds, after a similar analysis,

$$\langle T^{\mu\nu}(x) T^{\rho\sigma}(0)\rangle = \frac{c}{|x|^{2d}} \left(\frac{1}{2}(I^{\mu\sigma}(x) I^{\nu\rho}(x) + I^{\mu\rho}(x) I^{\nu\sigma}(x)) - \frac{1}{d}\eta^{\mu\nu}\eta^{\sigma\rho} \right) . \tag{3.58}$$

The conservation condition $\partial_\mu \langle T^{\mu\nu}(x) T^{\rho\sigma}(0)\rangle = 0$ fixes the dimension $\Delta = d$, which again makes sense from a dimensional analysis point of view. The component T^{00} is the energy density, which has units of mass per unit volume, or in our relativistic field theory framework where $\hbar = c = 1$, dimensions of mass to the d power. Unlike the case of conformal primary operators, whose normalization can often be adjusted, the normalization of the two-point function of the stress tensor is a physical quantity. The stress-tensor is a composite operator, made up of a product of conformal primaries. It is thus secretly a higher point correlation function in a limit where some of the points are taken to be coincident and divergences subtracted. The number c is called the central charge. The numbers τ and c play an important role in characterizing CFTs.

3.3 Adding a Defect

Adding a defect separates operators into two classes – bulk and defect. As there is a preserved $SO(p, 2)$ conformal group, correlation functions involving defect operators follow the same rules that we discussed above. Two and three point functions are fixed up to constants, while four point functions depend on cross ratios. Once we add bulk operators, however, the story becomes much richer.

While one point functions must vanish in the absence of a defect (and also for defect operators), they no longer need vanish in the presence of a boundary or defect. One natural way of thinking about such an object is as a two-point function between the inserted local operator and the nonlocal defect. In the case of a boundary, another natural picture for building intuition is to think of the boundary as a mirror. Thus the one-point function is like a two-point function between the operator and its image on the other side of the boundary.

In CFT without a defect, there is no way to build a one point function consistent with translational and scaling symmetry. Translational symmetry means the one-point function must be a constant. Dimensional analysis indicates such a constant must be dimensionful, which violates the scaling symmetry. (The identity operator, with scaling dimension zero, is the one important exception to this rule.) Once we add a defect, we can build a one-point function using the radial distance to the defect. In general, such one point functions for scalar operators take the form

$$\langle O(x) \rangle = \frac{c_O}{|r|^{\Delta_O}} \ . \tag{3.59}$$

Let us verify that this expression obeys the Ward identity for the special conformal transformations that preserve the defect. The analog of the remarkable property $|x_1' - x_2'| = \gamma_1^{-1/2}\gamma_2^{-1/2}|x_1 - x_2|$ that we used before for two-point functions reduces to $|r'| = \gamma^{-1}|r|$. The corresponding factor of $\gamma^{-\Delta}$ then cancels against the factor of $\Omega^{\Delta/2} = \gamma^{\Delta}$ in the Ward identity (3.37).

While it turns out that only scalar operators get expectation values in boundary CFT, in defect CFT, certain tensor operators (most notably among them the stress tensor) can get expectation values as well. The tensor structure $J_{\mu\nu}$ that makes this exception possible only has support in the directions transverse to the defect. Thus $J_{ab} = J_{ai} = J_{ia} = 0$, using i and j to index these transverse directions and a and b for the tangential ones. The remaining transverse components have the form

$$J^{ij} = \delta^{ij} - \frac{r^i r^j}{|r|^2} \ . \tag{3.60}$$

This weight zero tensor is clearly invariant under the rotations and translations in the defect directions. Special conformal transformations are only slightly less trivial and follow from the rule $r^i \to r^i/\gamma$. So for example, provided the codimension $q > 1$, the stress tensor expectation value will be specified by a constant, often called h,

$$\langle T_{\mu\nu} \rangle = \frac{h}{|r|^d} \left(J_{\mu\nu} - \frac{q-1}{d} \eta_{\mu\nu} \right) \ , \tag{3.61}$$

where we have subtracted $\eta_{\mu\nu}$ with a particular coefficient to ensure that $\langle T^{\mu}_{\mu}\rangle = 0$. We let the reader verify that $\partial_\mu \langle T^{\mu\nu}\rangle = 0$ precisely for the the choice $|r|^d$ in the denominator and no other power law. Powers of this $J_{\mu\nu}$ tensor, appropriately trace subtracted, can be used to express one-point functions of operators in higher dimensional symmetric irreducible representations of the Lorentz group as well.

The analog of CFT three point functions in the defect CFT situation are bulk-defect two-point functions. The objects are again characterized by constants. There is a relatively rich story involving tensorial versions of these two-point functions, but let us restrict to the scalar setting for now, where we put a hat on an operator to indicate that it lives on the defect:

$$\langle O_1(x)\hat{O}_2(0)\rangle = \frac{C_{12}}{|r|^\alpha |x|^{\Delta_1+\Delta_2-\alpha}} \ . \tag{3.62}$$

We made the split $x = (\mathbf{x}, r)$ into tangential and transverse directions and put $\hat{O}$ at the origin using translation invariance. The over-all power in the denominator is fixed by dimensional analysis, and we will now be able to fix α using special conformal transformations. Only c_{12} is unfixed by symmetry. We have

$$\frac{C_{12}}{|r|^\alpha |x|^{\Delta_1+\Delta_2-\alpha}} = \frac{1}{\gamma_1^{\Delta_1}\gamma_2^{\Delta_2}} \frac{C_{12}}{|r'|^\alpha |x'|^{\Delta_1+\Delta_2-\alpha}} = \frac{\gamma_1^\alpha (\gamma_1\gamma_2)^{\frac{\Delta_1+\Delta_2-\alpha}{2}}}{\gamma_1^{\Delta_1}\gamma_2^{\Delta_2}} \frac{C_{12}}{|r|^\alpha |x|^{\Delta_1+\Delta_2-\alpha}} \tag{3.63}$$

where in the first equality we used the Ward identity (3.37) and in the second the transformation properties of r and x under special conformal transformations that preserve the defect. In order for the equality to hold, we find $\alpha = \Delta_1 - \Delta_2$ and hence

$$\langle O_1(x)\hat{O}_2(0)\rangle = \frac{C_{12}}{|r|^{\Delta_1-\Delta_2} |x|^{2\Delta_2}} \ . \tag{3.64}$$

From the intuition about mirror charges, two bulk operators in a defect theory should be roughly equivalent to four insertions in a theory without defects. Indeed, in this case, similar to the four point function discussed above, we are able to construct two invariant cross ratios:

$$\xi_1 = \frac{(x_1-x_2)^2}{4|r_1||r_2|} \ , \quad \xi_2 = \frac{r_1\cdot r_2}{|r_1||r_2|} \ . \tag{3.65}$$

The second cross ratio ξ_2 is the cosine of an equatorial angle around the defect and reduces to $\xi_2 = 1$ for a boundary or interface. The two point function can be written in the form

$$\langle O_1(x_1) O_2(x_2)\rangle = \frac{f(\xi_1,\xi_2)}{|r_1|^{\Delta_1}|r_2|^{\Delta_2}} \ . \tag{3.66}$$

We will let the reader entertain themselves by considering more general types of correlation functions, for example defect-defect-bulk three point functions or bulk-bulk two point functions in nontrivial representations of the Lorentz group.

4 Conformal Symmetry in Curved Space

The conformal symmetry group is complicated, and it is often valuable to try to find conceptually more efficient ways of representing it. One method which we shall not touch in these lectures is the embedding space or null-cone formalism, where one uplifts the d-dimensional CFT to $d+2$ dimensions where $SO(d,2)$ acts linearly, as matrix multiplication. Another is to think about d-dimensional flat space as a limit of the QFT in curved space. It is often a surprisingly simple exercise to write a flat space space action in a diffeomorphism invariant form in curved space. For example, the massless scalar field in flat space

$$S = -\frac{1}{2}\int \mathrm{d}^d x (\partial_\mu \phi)(\partial^\mu \phi) \ , \tag{4.1}$$

where indices are raised and lowered with the Minkowski tensor $\eta_{\mu\nu}$, becomes

$$S = -\frac{1}{2}\int \mathrm{d}^d x \sqrt{-g}(\partial_\mu \phi)(\partial^\mu \phi) \tag{4.2}$$

in curved space where now indices are raised and lowered with the full metric tensor $g_{\mu\nu}$ and $\sqrt{-g}$ is shorthand for $\sqrt{-\det(g_{\mu\nu})}$. Such a naive approach will miss terms that vanish in flat space, for example $R\phi^2$ where R is the Ricci scalar, which can turn out to be very important.

Conformal symmetry from the curved space perspective are the set of diffeomorphisms which leave the metric invariant up to rescaling by a local function:

$$g_{\mu\nu} \to g'_{\mu\nu} = \frac{\partial x^\alpha}{\partial x'^\mu}\frac{\partial x^\beta}{\partial x'^\nu} g_{\alpha\beta} = \Omega(x) g_{\mu\nu} \ . \tag{4.3}$$

As diffeomorphism is trivially a symmetry of the theory in curved space, what we require from this perspective for conformal symmetry is an additional symmetry under rescaling of the metric. The additional symmetry has a name – Weyl symmetry.

Problem 4.1. *In the case of the free scalar field, the simple $(\partial\phi)^2$ action is not Weyl symmetric. However, if one adds the $R\phi^2$ term*

$$S = -\frac{1}{2}\int \mathrm{d}^d x \sqrt{-g}\left[(\partial_\mu \phi)(\partial^\mu \phi) + \xi R \phi^2\right] \tag{4.4}$$

where $\xi = \frac{(d-2)}{4(d-1)}$, the action is Weyl symmetric. Verify this fact, assuming $\phi \to \Omega^{\frac{d-2}{4}}\phi$ and $g_{\mu\nu} \to \Omega^{-1} g_{\mu\nu}$ under Weyl rescaling.

A convenient side effect of moving to curved space is a simple method for computing the stress-energy tensor. We introduced this tensor, which describes the flow of energy and momentum, in the context of Noether's theorem and translation invariance. The stress-energy tensor is the conserved current associated with translation symmetry. However, an alternate definition is the response of the action to variation of the metric:

$$\delta S = \frac{1}{2}\int \mathrm{d}^d x \sqrt{-g} T^{\mu\nu} \delta g_{\mu\nu} \ . \tag{4.5}$$

Diffeomorphisms are symmetries for which infinitesimally $x^\mu \to x'^\mu = x^\mu + \epsilon^\mu(x)$. The metric changes infinitesimally as $\delta g_{\mu\nu} = -\nabla_\mu \epsilon_\nu - \nabla_\nu \epsilon_\mu$. That this transformation is a symmetry means that δS should vanish in this case. Integrating by parts, we conclude the stress-tensor is conserved $\nabla_\mu T^{\mu\nu} = 0$. If we also insist Weyl scaling $\delta g_{\mu\nu} = \lambda(x) g_{\mu\nu}$ is a symmetry, then we conclude that the stress-tensor is traceless, $T^\mu_{\ \mu} = 0$.[4,5] In an exercsie below, we leave it to the reader to verify that for a conformally coupled scalar field $\xi = \frac{(d-2)}{4(d-1)}$, the stress tensor that follows from varying the action (4.4) in the flat space limit agrees with the improved stress tensor (3.27) from the previous section.

Tracelessness of the stress tensor is an oft cited property of conformal field theories. In fact, Weyl symmetry is almost always anomalous. In other words it is a symmetry classically but spoiled by quantum effects, when we consider the full path integral for the field theory. One finds in general curved space-time that the trace of the stress tensor is proportional to a sum over curvature invariants with special properties. These "trace anomalies" feature prominently in the study of conformal field theory, and will be an important topic in these lectures. In 2d CFT, for example, $\langle T^\mu_{\ \mu} \rangle = \frac{c}{24\pi} R$ where R is the Ricci scalar curvature and c is a constant, the central charge of the CFT.

Problem 4.2. *Compute the stress tensor in the flat space limit $g_{\mu\nu} = \eta_{\mu\nu}$ for the scalar field of problem 4.1 with the conformal coupling $\xi = \frac{(d-2)}{4(d-1)}$. Check that $T^{\mu\nu}$ is conserved and traceless on-shell in the flat space limit.*

Of course in these lectures, we are also concerned about what happens when there is a defect or boundary in the space-time. Let us parametrize the location of the defect in curved space with the coordinates χ^a, $a = 1, \ldots, p$. We then have a vector of embedding functions $x^\mu = X^\mu(\chi^a)$ telling us where the defect is located. Above, we saw that the stress-tensor was the operator that parametrized the response of the system to changes in the metric. There is also an operator which parametrizes the response of the system to changes in the location of the defect. This operator is traditionally called the displacement operator. More generally then, (4.5) becomes

$$\delta S = \frac{1}{2} \int_M \mathrm{d}^d x \sqrt{-g} T^{\mu\nu} \delta g_{\mu\nu} - \int_N \mathrm{d}^p x \sqrt{-\gamma} D_\mu \delta X^\mu \ . \tag{4.6}$$

We have introduced here a manifold M and N a p-dimensional sub-manifold with defect metric γ_{ab} induced from the bulk metric via the embedding $X^\mu(\chi^a)$. In this more general setting, bulk diffeomorphism invariance also acts on the defect, $\delta X^\mu = \epsilon^\mu$, spoiling conservation of the stress tensor. Let us specialize to the situation where the defect is simply $\mathbb{R}^{p-1,1} \subset \mathbb{R}^{d-1,1}$. In this limit

$$\partial_\mu T^{\mu\nu} = \delta^{(q)}(x^i) D^\nu \tag{4.7}$$

[4]Note that global scale invariance, where λ is a constant, is not enough to guarantee tracelessness. It only guarantees that $T^\mu_{\ \mu}$ is a total derivative. The special conformal transformations, where λ depends on x, are needed to guarantee tracelessness.

[5]We are playing a little fast and loose here. In analyzing the transformation of the action with respect to the symmetry, either Weyl rescaling or diffeomorphism, the fields will transform as well. However, these conditions on the stress tensor are expected to hold only on-shell, after applying the equations of motion. The equations of motion are derived by varying the action with respect to the fields. Thus the equations of motion can be used to zero out the contribution from varying the fields in computing δS for the symmetry transformation, leaving only the contribution from $\delta g_{\mu\nu}$.

where $\delta^{(q)}(x^i)$ is a delta function restricting to the $\mathbb{R}^{p-1,1}$ defect.

We have one additional symmetry to consider, diffeomorphism invariance of the coordinates χ^a describing the defect. Note that such a diffeo $\chi^a \to \chi^a + \varepsilon^a(\chi)$ will change the embedding function

$$\delta X^\mu = \varepsilon^a \partial_a X^\mu \tag{4.8}$$

which in turn implies the vanishing of $\varepsilon^a \partial_a X^\mu D_\mu = 0$. Since $\varepsilon^a \partial_a X^\mu$ is tangent to the boundary, the only non-vanishing components of D^μ are perpendicular. In the particular case of a boundary $q = 1$, for example, the displacement operator is not a vector. It's a scalar. Moreover, a Gauss law type argument shows that $\lim_{x^n \to 0} T^{nn} = D^n$.

4.1 More Ward Identities

These connections between the metric $g_{\mu\nu}$ and the stress-tensor $T^{\mu\nu}$ and between the embedding map X^μ and the displacement operator D^i allow us to use the conformal symmetry to constrain correlation functions involving $T^{\mu\nu}$ and D^i. The strategy is to realize that a conformal variation of a correlation function, written schematically as $\langle \delta \mathcal{X} \rangle$, can be related to a metric variation or embedding map variation of that same correlation function and thus to the correlation function with an extra insertion of the stress-tensor $\langle T_{\mu\nu} \mathcal{X} \rangle$ or displacement operator $\langle D^i \mathcal{X} \rangle$.

Let us focus first on conformal transformations that preserve the defect, if indeed a defect is present. Thus, we will find relations between $\langle \delta \mathcal{X} \rangle$ and $\langle T_{\mu\nu} \mathcal{X} \rangle$ as the embedding map X^μ is left invariant by such transformations. The first step is to understand in more detail how $\langle \mathcal{X} \rangle$ transforms. For a combination diffeomorphsim plus Weyl transformation, we have

$$\delta x^\mu = \epsilon^\mu \ , \quad \delta g_{\mu\nu} = 2\omega g_{\mu\nu} - \nabla_\mu \epsilon_\nu - \nabla_\mu \epsilon_\nu \ , \tag{4.9}$$

Expanding for a linearized transformation, we have then that

$$\begin{aligned} \langle \delta \mathcal{X} \rangle_g &= \langle \mathcal{X} \rangle_{g+\delta g} - \langle \mathcal{X} \rangle_g \\ &= \frac{1}{2} \int_M \mathrm{d}^d x \sqrt{-g} \langle T^{\mu\nu} \mathcal{X} \rangle_g \delta g_{\mu\nu} \\ &= \int_M \mathrm{d}^d x \sqrt{-g} (\omega g_{\mu\nu} + \epsilon_\mu \nabla_\nu) \langle T^{\mu\nu} \mathcal{X} \rangle_g \ , \end{aligned} \tag{4.10}$$

For ϵ^μ and ω corresponding to a conformal transformation, $\delta g_{\mu\nu} = 0$ and the variation of the correlation function should vanish, in accord with conformal symmetry. In section 3, we considered the finite version rather than the infinitesimal version of this constraint to restrict the form of two and three point functions. The energetic reader is urged to check that the form we deduced is correct and that, for example, δ acting on a two-point function vanishes.

To get a less trivial constraint, we will instead focus on cases where $\omega \neq 0$ and $\epsilon^\mu \neq 0$ only in some region $B \subset M$ of space-time that surrounds some of the operators in $\mathcal{X}$. Consider applying Stokes' Theorem to the integral of the stress-tensor on the surface ∂B bounding this region:

$$\begin{aligned}\int_{\partial B} \mathrm{d}\sigma^\mu \epsilon^\nu \langle T_{\mu\nu} \mathcal{X} \rangle_g &= \int_B \mathrm{d}^d x \sqrt{-g} \nabla^\mu \langle \epsilon^\nu T_{\mu\nu} \mathcal{X} \rangle_g \\ &= \int_B \mathrm{d}^d x \sqrt{-g} \langle (\omega g^{\mu\nu} T_{\mu\nu} + \epsilon^\nu \nabla^\mu T_{\mu\nu}) \mathcal{X} \rangle_g \\ &= \langle \delta|_B \mathcal{X} \rangle_g \ ,\end{aligned} \tag{4.11}$$

where we denoted by $\delta|_B$ the variation δ acting only inside the domain B. In the second line, we restricted to a conformal type variation of $g_{\mu\nu}$, albeit restricted to B, and in the last line, we used our result (4.10) from above. This innocent looking integral relation is actually quite powerful. As a first exercise, one can consider the boundary ∂B to be the slice $t = 0$ and B to be the region $t > 0$. If we have $\mathcal{X} = \mathcal{X}_{t<0} \mathcal{X}_{t>0}$, then indeed integrating the $\epsilon_\mu T^{\mu 0}$ over the surface ∂B will give the different conformal generators P^μ, D, K^μ, or rotations, depending on the choice ϵ_μ.

Another interesting constraint that comes from (4.11) is a relation between $\langle O(x_1) O(x_2) \rangle$ and $\langle T_{\mu\nu}(x) O(x_1) O(x_2) \rangle$ for a primary operator $O(x)$ with scaling dimension Δ. The idea is to put $x_2 \to \infty$ and $x_1 = 0$ and choose the region B to be a ball of radius one centered about the origin. To see how this works in more detail, however, we first need to understand how conformal symmetry and conservation fix $\langle T_{\mu\nu}(x) O(x_1) O(x_2) \rangle$ up to an overall constant. The three point function can be constructed from the vector

$$V^\mu = \frac{x_1^\mu - x^\mu}{|x_1 - x|^2} - \frac{x_2^\mu - x^\mu}{|x_2 - x|^2} \ , \tag{4.12}$$

normalized to have unit lenth $\hat{V} = V/|V|$. One way of understanding why this vector structure has the correct transformation properties is that it appears as a derivative of a cross ratio, and in fact conveniently only three of the four points in the cross ratio show up in the derivative. The form is

$$\langle T_{\mu\nu}(x) O(x_1) O(x_2) \rangle = c_{TOO} \frac{\hat{V}_\mu \hat{V}_\nu - \frac{1}{d}\eta_{\mu\nu}}{|x - x_1|^d |x - x_2|^d |x_1 - x_2|^{2\Delta - d}} \ . \tag{4.13}$$

Integrating the three point function over the unit sphere for a dilatation then yields the relation

$$c_{TOO} \left(\frac{1}{d} - 1 \right) \mathrm{Vol}(S^{d-1}) = \Delta c_{OO} \ , \tag{4.14}$$

where $\mathrm{Vol}(S^{d-1})$ is the volume of a sphere of unit radius.

Our main interest in these lectures are cases with boundaries and defects, and so let us consider a slightly more elaborate example, involving the two point function $\langle T_{\mu\nu}(x_1) O(x_2) \rangle$ in the boundary case. Naively, conformal invariance alone means the two point function will involve a function of a cross ratio. However, conservation fixes this expression up to an overall constant

$$\langle T_{\mu\nu}(x) O(x') \rangle = c_{TO} \left[16 \xi_1 (1 + \xi_1) \right]^{-\frac{d}{2}} \frac{\hat{V}_\mu \hat{V}_\nu - \frac{1}{d}\eta_{\mu\nu}}{|r|^d |r'|^\Delta} \ . \tag{4.15}$$

The weight zero vector is analogous to the V_μ from the previous example, where the third point is the mirror image of x', sitting at $(\mathbf{x}', -r)$. The prefactor $[16\xi_1(1+\xi_1)]$ can similarly be understood, and the 16 is included to soak up the factors of 2 in the definition of ξ_1 (3.65). Dilatations produce, in a calculation now very similar to that which produced (4.14), the relation

$$\frac{c_{TO}}{2^d}\left(\frac{1}{d}-1\right)\mathrm{Vol}(S^{d-1}) = \Delta c_O \ . \tag{4.16}$$

There is an important conceptual difference from the previous example, however. We are applying the dilatation at the point x', which is not in fact a symmetry of the theory. Only dilatations centered on the boundary (or defect) are preserved. Thus in order for D acting on O to produce a Δ, we need to assume that $O(x)$ is a primary operator not just with respect to the residual conformal group preserved by the boundary but with respect to the full conformal group before the addition of the boundary.

Let us now broaden our scope to investigate transformations that affect the location of the defect $X^i(\chi^a)$ as well as the metric $g_{\mu\nu}$. The analog of (4.10) is

$$\begin{aligned}\langle\delta\mathcal{X}\rangle &= \frac{1}{2}\int_M \mathrm{d}^d x\sqrt{-g}\langle T^{\mu\nu}\mathcal{X}\rangle\delta g_{\mu\nu} - \int_N \mathrm{d}^p x\,\sqrt{-\gamma}\langle D^i\mathcal{X}\rangle\delta X_i \\ &= \int_M \mathrm{d}^d x\sqrt{-g}(\omega g_{\mu\nu}+\epsilon_\mu\nabla_\nu)\langle T^{\mu\nu}\mathcal{X}\rangle - \int_N \mathrm{d}^p x\,\sqrt{-\gamma}\,\epsilon_i\langle D^i\mathcal{X}\rangle \ .\end{aligned} \tag{4.17}$$

where γ_{ab} is the induced metric on the defect. The Ward identity (3.37) then gets an additional contribution from the displacement operator

$$\int_{\partial B}\mathrm{d}\sigma^\mu\epsilon^\nu\langle T_{\mu\nu}\mathcal{X}\rangle - \int_{N\cap B}\mathrm{d}^p x\,\sqrt{-\gamma}\,\epsilon_i\langle D^i\mathcal{X}\rangle = \langle\delta|_B\mathcal{X}\rangle \ . \tag{4.18}$$

The boundary case $q=1$ requires a little more care: In the case that $N\cap B$ is not empty, we can either include the boundary in ∂B and discard the displacement term or treat the boundary as separate and interpret $D^i = T^{nn}$.

As an example, we can use this Ward identity in the case of a translation perpendicular to the defect to compare $\langle O(x)\rangle = \frac{c_O}{|r|^\Delta}$ with

$$\langle O(x)D^i(x')\rangle = \frac{c_{DO}}{|r|^{\Delta-p-1}|x-x'|^{2p+2}} \ , \tag{4.19}$$

noting that the scaling dimension of D^i is $p+1$. We choose B to be all of space-time except the defect and drop the integral over the stress tensor. We need the following volume integral

$$\int\frac{\mathrm{d}^p x}{(x^2+r^2)^{2p+2}} = \frac{\mathrm{Vol}(S^p)}{2^{p+1}}\frac{1}{r^{p+2}} \ . \tag{4.20}$$

Noting that $\epsilon^\mu \sim r$, we find that

$$\frac{c_{DO}\,\mathrm{Vol}(S^p)}{2^{p+1}} = \Delta c_O \tag{4.21}$$

which agrees with the result (4.16) in the special case $p = d - 1$.

One could go on and obtain further relations between correlation functions. For example, in the defect context, the stress tensor correlation functions $\langle T_{\mu\nu} \rangle$ and $\langle T_{\mu\nu} D^i \rangle$ are fixed up to constants. The above Ward identity will fix a relation between these constants.[6] Similarly, since $\langle J_\mu \rangle$ will vanish for any bulk vector operator, so will $\langle J_\mu D^i \rangle$. In the cases where correlation functions are fixed only up to functions of invariant cross ratios, these Ward identities will relate integrals of higher point functions involving $T_{\mu\nu}$ and D^i to lower points functions without these insertions.[7]

5 Radial Quantization and the Operator Product Expansion

In introducing the notion of a conformal primary state $|\phi_I\rangle$ and conformal primary operator $\phi_I(x)$ in the previous chapter, the origin played a special role: $|\phi_I\rangle = \phi_I(0)|0\rangle$. The origin plays such a role because in defining the dilatation operator on function space, $D = x^\mu \partial_\mu$, we chose to think about it as scale transformations with respect to the origin. (Of course, we could equally well have chosen to dilate space about some other point $\hat{D}(x) = e^{-iPx} D e^{iPx}$.)

There is a different and useful way of thinking about the origin. Let's instead return to the standard QFT framework, where we can create in and out states by acting on the vacuum in the far past and far future, $|\psi_{\rm in}\rangle = \lim_{t\to-\infty} \hat{\psi}(t)|0\rangle$ and $|\psi_{\rm out}\rangle = \lim_{t\to\infty} \hat{\psi}(t)|0\rangle$, with some local operator $\hat{\psi}(t)$.

In a conformal field theory, in a Euclidean context where all the coordinates are spatial, people often choose to think about the radial coordinate as a time-like coordinate. Suppose we write the line element of flat space as a foliation of spheres

$$ds^2 = dr^2 + r^2 d\Omega^2 \tag{5.1}$$

where $d\Omega^2$ is the line element on a $(d-1)$-dimensional sphere of unit radius and $r > 0$. Then we could equally well decide to define a new radial coordinate $r = e^\tau$ in which case the line element becomes

$$ds^2 = e^{2\tau} \left(d\tau^2 + d\Omega^2 \right) . \tag{5.2}$$

In the new coordinate system τ ranges from $-\infty < \tau < \infty$. We can think about the point $r = 0$ as the far past. Similarly $r \to \infty$ is the far future.

We mentioned before that in a CFT context, the dilatation operator D largely replaces the Hamiltonian P^0. While in QFT, we can use the evolution operator $U(t) = e^{itP^0}$ to move from time slice to time slice, in a CFT framework, we can use instead the operator $U(r) = e^{i\tau D}$ to move from radial slice to radial slice. In QFT, we have $P^0 = -i\partial_t$. In CFT, on the other hand, we have $D = -ix^\mu \partial_\mu = -ir\partial_r = -i\partial_\tau$.

[6] M. Billò, V. Gonçalves, E. Lauria and M. Meineri, "Defects in conformal field theory," JHEP **04**, 091 (2016) [arXiv:1601.02883 [hep-th]].

[7] C. P. Herzog and V. Schaub, "A Sum Rule for Boundary Contributions to the Trace Anomaly," [arXiv:2107.11604 [hep-th]].

From this point of view of "radial quantization", the conformal primary state $|\phi_I\rangle$, created at the origin by $\phi_I(0)$, can be thought of as a standard in-state in a usual QFT context. Similarly, there are out states which are created by inserting operators at large radial distance from the origin.

There are some technical perils in this program which we will not dwell on overly. The first is that $e^{i\tau D}$ is not unitary. Another is how exactly to define a useful inner product with the out states.

5.1 Operator Product Expansion

The next exercise is to consider the state

$$|\psi\rangle = \phi_1(x)\phi_2(0)|0\rangle \ . \tag{5.3}$$

For simplicity, we can consider the case where both operators are scalars. Because ψ is a state and because the space of states is spanned by eigenstates of the dilatation operator, we can expand ψ in a basis of such states:

$$|\psi\rangle = \sum_n c_n |\Delta_n\rangle \ . \tag{5.4}$$

Moreover, we know that these eigenstates come in multiplets. Each multiplet contains a conformal primary state $|\phi_I\rangle$ and its descendants $P^{\mu_1}\cdots P^{\mu_n}|\phi_I\rangle$. We can therefore write the state $|\psi\rangle$ in the form

$$\phi_1(x)\phi_2(0)|0\rangle = \sum_{\phi_I} C_{\Delta,I}(x,\partial)\phi_I(0)|0\rangle \ , \tag{5.5}$$

where we will discuss the precise form of $C_{\Delta,I}(x,\partial)$ momentarily. Here Δ is the scaling dimension of ϕ_I.

In fact, we can promote this operator product expansion from a discussion of states to a discussion of the operator algebra itself:

$$\phi_1(x)\phi_2(0) = \sum_{\phi_I} C_{\Delta,I}(x,\partial)\phi_I(0) \ , \tag{5.6}$$

where implicitly the equality holds only inside correlation functions, and also only where the additional operators inside the correlation function are inserted outside the sphere, centered at the origin, of radius $|x|$. Said another way, the insertion of a third operator $\phi(x')$ in the correlation function $\langle\phi(x')\phi_1(x)\phi_2(0)\rangle$ sets a radius of convergence for the small x expansion, namely $|x| < |x'|$.

Now let us try to pin down the form of $C_{\Delta,I}$. By dimensional analysis, for a scalar operator $\phi(x)$ of dimension Δ, we can see that

$$C_\Delta(x,\partial)\phi(0) = \frac{c}{|x|^{\Delta_1+\Delta_2-\Delta}}(\phi(0)+\ldots) \ . \tag{5.7}$$

The ellipsis refers to all of the descendants of $\phi(x)$. To check this guess, we can act with the dilatation operator. Acting on the left hand side of (5.6) yields

$$\begin{aligned} D\phi_1(x)\phi_2(0)|0\rangle &= i(\Delta_1 + x^\mu\partial_\mu)\phi_1(x)\phi_2(0)|0\rangle + i\Delta_2\phi_1(x)\phi_2(0)|0\rangle \\ &= i(\Delta_1+\Delta_2)\phi_1(x)\phi_2(0)|0\rangle + x^\mu\partial_\mu\phi_1(x)\phi_2(0)|0\rangle \ . \end{aligned} \tag{5.8}$$

We now substitute the guess (5.7) for $\phi_1(x)\phi_2(0)$ in the second term, focusing on the contribution of dimension Δ to this operator product expansion:

$$D\phi_1(x)\phi_2(0)|0\rangle \sim i(\Delta_1+\Delta_2-(\Delta_1+\Delta_2-\Delta))\frac{c}{|x|^{\Delta_1+\Delta_2-\Delta}}(\phi(0)+\ldots)|0\rangle \ . \tag{5.9}$$

Acting directly on the right hand side of (5.7) with D yields the same result to leading order in a small x expansion:

$$D\frac{c}{|x|^{\Delta_1+\Delta_2-\Delta}}(\phi(0)+\ldots)|0\rangle = i\Delta\frac{c}{|x|^{\Delta_1+\Delta_2-\Delta}}(\phi(0)+\ldots)|0\rangle \ . \tag{5.10}$$

Problem 5.1. *Continuing the small $|x|$ expansion of $C_\Delta(x,\partial)$, we find at next order*

$$C_\Delta(x,\partial)\phi(0) = \frac{c}{|x|^{\Delta_1+\Delta_2-\Delta}}\left(1+\alpha x^\mu\partial_\mu+\ldots\right)\phi(0) \ .$$

By acting with K^μ on boths sides, show that $\alpha = \frac{\Delta_1-\Delta_2+\Delta}{2\Delta}$.

In fact conformal invariance completely fixes the form of $C_{\Delta,I}(x,\partial)$ up to an overall constant, which we called c in the discussion above. A more efficient way to compute $C_\Delta(x,\partial)$ is as follows. Consider expanding the following three point function of three scalar operators using the operator product expansion

$$\langle\phi_1(x)\phi_2(0)\phi_\Delta(z)\rangle = \sum_{\Delta'} C_{12\Delta'}C_{\Delta'}(x,\partial_y)\langle\phi_{\Delta'}(y)\phi_\Delta(z)\rangle_{y=0} \ . \tag{5.11}$$

The constant c has now been renamed $C_{12\Delta'}$ and pulled out of the definition of $C_\Delta(x,\partial)$. All the higher spin primaries in the operator product expansion will have vanishing expectation value with ϕ_Δ and so we can restrict the sum to scalar primaries. In fact only scalar primaries with dimension $\Delta' = \Delta$ will contribute to the sum:

$$\langle\phi_1(x)\phi_2(0)\phi_\Delta(z)\rangle = C_{12\Delta}C_\Delta(x,\partial)\langle\phi_\Delta(y)\phi_\Delta(z)\rangle_{y=0} \ . \tag{5.12}$$

Conformal invariance forces the two and three point functions to have the form

$$\langle\phi_\Delta(y)\phi_\Delta(z)\rangle = \frac{1}{|y-z|^{2\Delta}} \ , \tag{5.13}$$

$$\langle\phi_1(x)\phi_2(0)\phi_\Delta(z)\rangle = \frac{C_{12\Delta}}{|x|^{\Delta_1+\Delta_2-\Delta}|z|^{\Delta_2+\Delta-\Delta_1}|x-z|^{\Delta_1+\Delta-\Delta_2}} \ . \tag{5.14}$$

where we have taken the liberty of fixing the normalization of the two-point function in a conventional CFT way. By expanding out the left hand side of (5.12) for small $|x|$ and matching to the right hand side, we can fix the form of $C_\Delta(x,\partial)$. Note that having normalized the two-point function to unity, the constant $C_{12\Delta}$ in the operator product expansion and in the three point function are naturally identified, fixing a normalization for $C_\Delta(x,\partial)$.

Problem 5.2. *Use this procedure to compute the first three terms in $C_\Delta(x,\partial)$.*

5.2 Conformal Blocks

We now apply this notion of the operator product expansion to higher point functions. For simplicity, let us consider the correlation function of four identical scalar primaries $\Phi(x)$ with dimension η. From the discussion at the end of section 3, we saw that conformal symmetry constrains the four point correlation function to have the form

$$\langle \Phi(x_1)\Phi(x_2)\Phi(x_3)\Phi(x_4)\rangle = \frac{G(u,v)}{|x_{12}|^{2\eta}|x_{34}|^{2\eta}} \tag{5.15}$$

where u and v were the invariant cross ratios (3.53) formed from combinations of the differences x_{ij} between the insertion locations.

Given the technology of the operator product expansion, we can take x_1 close to x_2 and x_3 close to x_4 and write pairs of the operators in the four point function as sums over conformal primaries:

$$\Phi(x_1)\Phi(x_2) = \sum_{\Delta,I} c_{\Delta,I} C_{\Delta,I}(x_{12},\partial_y)\phi_{\Delta,I}(y)\big|_{y=x_2} \ , \tag{5.16}$$

$$\Phi(x_3)\Phi(x_4) = \sum_{\Delta,I} c_{\Delta,I} C_{\Delta,I}(x_{34},\partial_z)\phi_{\Delta,I}(z)\big|_{z=x_4} \ . \tag{5.17}$$

The $c_{\Delta,I}$ are the OPE coefficients, or equivalently the coefficients in the three point functions if we normalize the two-point functions in the conventional way. Inserting these decompositions into the four point function, we obtain the sum (see fig. 2a)

$$\langle \Phi(x_1)\Phi(x_2)\Phi(x_3)\Phi(x_4)\rangle = \sum_{\Delta,I} c^2_{\Delta,I}\left[C_{\Delta,I}(x_{12},\partial_y)C_{\Delta,I}(x_{34},\partial_z)\langle\phi_{\Delta,I}(y)\phi_{\Delta,I}(z)\rangle\right]\big|_{y=x_2,z=x_4} \tag{5.18}$$

Note the double sum collapses to a single sum because the two point function between two conformal primaries vanishes unless the operators have the same conformal dimension and spin.

The important point here is that the term in brackets is completely fixed by conformal invariance. By convention, we define a *conformal block* $G_{\Delta,I}(u,v)$ such that

$$\left[C_{\Delta,I}(x_{12},\partial_y)C_{\Delta,I}(x_{34},\partial_z)\langle\phi_{\Delta,I}(y)\phi_{\Delta,I}(z)\rangle\right]\big|_{y=x_2,z=x_4} = \frac{G_{\Delta,I}(u,v)}{|x_{12}|^{2\eta}|x_{34}|^{2\eta}} \ . \tag{5.19}$$

The conformal block is defined in a theory independent fashion by a choice of Lorentz representation I and conformal dimensions η and Δ. The theory dependent data in the four point function reduces to the operator product coefficients $c_{\Delta,I}$ and the conformal dimensions Δ.

A similar story holds true for higher point functions as well (see fig. 2). By bringing the insertions close together pairwise, one can decompose an arbitrary correlation function into a sum over conformal blocks. One can make thus a stronger statement that a conformal field theory is defined by the data – the spin and scaling dimension – of its conformal primaries along with the coefficients in their three point functions. With those in hand, one can

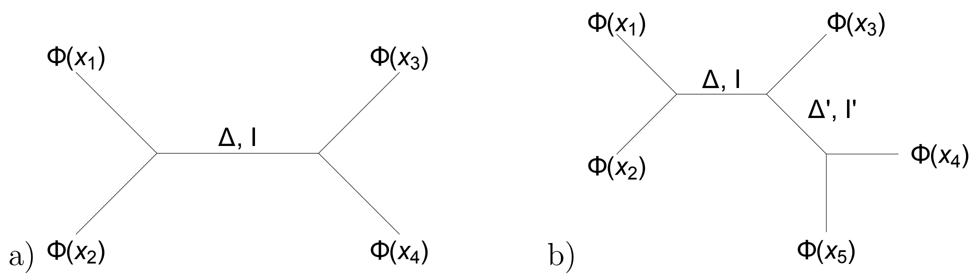

Figure 2: The decomposition of a) a four-point function and b) a five-point function into a sum over conformal blocks.

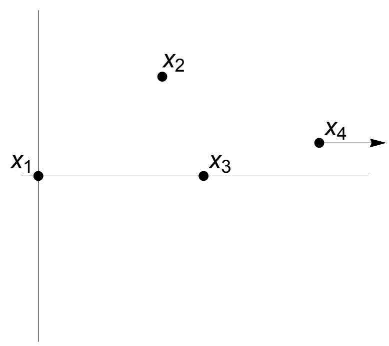

Figure 3: A useful configuration for understanding the z and $\bar{z}$ cross ratios.

reconstruct any correlation function in a conformal partial wave decomposition. In the case of the four point function, we can write

$$G(u,v) = \sum_{\Delta,I} c_{\Delta,I}^2 G_{\Delta,I}(u,v) \ . \tag{5.20}$$

We will see in the next section how to further constrain the operator spectrum and OPE coefficients that define a CFT by examining a particular constraint on this sum.

To be more concrete, we can give $G_{\Delta,I}$ for four identical scalars in four dimensions:

$$G_{\Delta,\ell}(z,\bar{z}) = \frac{z\bar{z}}{z-\bar{z}}(k_{\Delta+\ell}(z)k_{\Delta-\ell-2}(\bar{z}) - (z \leftrightarrow \bar{z}) \ , \tag{5.21}$$

where we have defined

$$k_\beta(z) = z^{\frac{\beta}{2}} {}_2F_1\left(\frac{\beta}{2}, \frac{\beta}{2}, \beta, z\right) \ .$$

We have also introduced $u = z\bar{z}$ and $v = (1-z)(1-\bar{z})$. To understand these new coordinates geometrically, one can place $x_1 = (0,0,\ldots)$, $x_3 = (1,0,0,\ldots)$ and x_4 at infinity. Then rotate the coordinate system to put x_2 in the xy-plane. The z coordinate is x_2, thinking of the xy-plane as a complex coordinate system (see fig. 3).

Problem 5.3. *By explicitly computing the first few terms in a small z expansion, verify the form of the conformal block for $\ell = 0$ and $d = 4$ by comparing it against your previous small x expansion of $C_\Delta(x, \partial)$.*

5.3 Deriving the Conformal Blocks

One method for deriving the expression (5.21) for the conformal blocks is to find a differential equation satisfied by $G_{\Delta,I}(u,v)$ and solve it. The claim is that $G_{\Delta,I}(u,v)$ is an eigenvector of the Casimir operator for the conformal group.

What is the Casimir operator? You have seen this object for the SO(3) rotation group in quantum mechanics. In that case the Casimir operator was also called J^2 and it was equal to the sum $J^2 = J_x^2 + J_y^2 + J_z^2$. This operator had eigenvalues $\ell(\ell+1)$ for a $2\ell+1$ dimensional representation of SO(3). More generally for a rotation (or Lorentz) group, we can write

$$\text{cas} = \frac{1}{2} M_{\mu\nu} M^{\mu\nu} \ . \tag{5.22}$$

In the case of SO(3), we have the relations $J_x = M_{yz}$, $J_y = M_{zx}$, and $J_z = M_{xy}$. The claim is that $[\text{cas}, M_{\mu\nu}] = 0$. Therefore everything in the same irreducible representation of the group will have the same eigenvalue with respect to the action of the Casimir operator.

In problem 2.7, we saw that the conformal group was also a rotation group, in particular the group $SO(d+1,1)$ (in the Euclidean case), with the identifications

$$M_{-10} = D \ , \quad M_{0i} = \frac{P_i + K_i}{2} \ , \quad M_{-1i} = \frac{P_i - K_i}{2} \tag{5.23}$$

with the metric $\eta_{-1,-1} = -1$ and $\eta_{00} = \eta_{ii} = 1$. The remaining elements M_{ij} are the generators of the usual Lorentz (or rotation) group inside the conformal group.

If we expand the Casimir operator out in terms of our more familiar P_i and K_j, we find that

$$\begin{aligned} \text{cas} &= \frac{1}{2} M^{\mu\nu} M_{\mu\nu} \\ &= \frac{1}{2} M^{ij} M_{ij} - D^2 + \frac{1}{2} P_i K^i + \frac{1}{2} K_i P^i \\ &= \frac{1}{2} M^{ij} M_{ij} - D(D - i\,d) - P_i K_i \ , \end{aligned} \tag{5.24}$$

where in the second line, I used the commutator $[K_i, P_j]$. We now apply this object to a primary state $|\phi_{\Delta,I}\rangle$ in order to learn its eigenvalue. (Note $i\,d$ is $\sqrt{-1}$ times the dimension, not the identity matrix.) For simplicity, let us assume that $\phi_{\Delta,I}$ transforms in a symmetric, traceless representation of the M_{ij} with spin ℓ. The claim is that

$$\text{cas}|\phi_{\Delta,I}\rangle = [\ell(\ell+d-2) + \Delta(\Delta-d)]\,|\phi_{\Delta,I}\rangle \ . \tag{5.25}$$

The first part of the eigenvalue $\ell(\ell+d-2)$ is the generalization of the $\ell(\ell+1)$ result for the $SO(3)$ group. The second term $\Delta(\Delta-d)$ can be read off by acting with D on $|\phi_{\Delta,I}\rangle$.

We are now ready to return to the question of conformal blocks for the four point function $\langle\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\rangle$ of four identical scalar operators. Let us insert a resolution of the identity into the four point function:

$$\langle\Phi(x_1)\Phi(x_2)\Phi(x_3)\Phi(x_4)\rangle = \sum_\psi \langle\Phi(x_1)\Phi(x_2)|\psi\rangle\langle\psi|\Phi(x_3)\Phi(x_4)\rangle \ . \tag{5.26}$$

We then restrict the sum to $|\phi_{\Delta,I}\rangle$ and its descendants, i.e. a representation of the conformal group, every member of which will have the same eiegenvalue with respect to the action of the Casimir operator. This restriction is by definition the contribution of one conformal block to the four point function:

$$\sum_{\psi}{}' \langle\Phi(x_1)\Phi(x_2)|\psi\rangle\langle\psi|\Phi(x_3)\Phi(x_4)\rangle = \frac{G_{\Delta,I}(u,v)}{|x_{12}|^{2\eta}|x_{34}|^{2\eta}} \ . \tag{5.27}$$

The claim is that $\text{cas}|\psi\rangle = [\ell(\ell+d-2)+\Delta(\Delta-d)]|\psi\rangle$ where $|\phi_{\Delta,I}\rangle$ is in a symmetric, traceless, spin ℓ representation of the Lorentz group and $|\psi\rangle$ in the multiplet with $|\phi_{\Delta,I}\rangle$. Inserting a Casimir operator and defining $\lambda_{\Delta,\ell} \equiv \ell(\ell+d-2)+\Delta(\Delta-d)$, we see that

$$\langle\Phi(x_1)\Phi(x_2)|\text{cas}|\psi\rangle = \lambda_{\Delta,\ell}\langle\Phi(x_1)\Phi(x_2)|\psi\rangle \ . \tag{5.28}$$

But we can also act with the Casimir operator to the left, using the representation of the conformal group on $\Phi(x_1)\Phi(x_2)$. (Of course, we could run the same argument with $\Phi(x_3)\Phi(x_4)$ as well, and will get the same answer.) When the dust settles, we find a second order, linear partial differential equation in the cross ratios u and v of the form

$$\text{cas}\, G_{\Delta,\ell}(u,v) = \lambda_{\Delta,\ell}G_{\Delta,\ell}(u,v) \ . \tag{5.29}$$

The solution to this differential equation are the conformal blocks of (5.21).

5.4 Adding a Defect or Boundary

In the presence of a boundary or defect, there are two types of operator product expansions that we may consider. The first is the one we considered above, where two local operators $\phi_1(x_1)$ and $\phi_2(x_2)$ get close together. That the OPE is the same one that we considered above requires nontrivial assumptions. The first is that the presence of a boundary or defect does not spoil a notion of local conformal invariance, that for example $\partial_\mu T^{\mu\nu} = 0$ away from the defect. The second is that these operators remain conformal primaries with respect to the original larger $SO(d+1,1)$ conformal symmetry group that was present in the absence of a defect. One could worry that a defect introduces for example some kind of power law field strengths that fall off as one moves away from the defect. We are assuming this kind of thing doesn't happen.

Recall that the two-point function in a defect theory is like a four-point function in a theory without defects and will generally depend on nontrivial functions of cross-ratios. This bulk OPE of $\phi_2(x_1)$ and $\phi_2(x_2)$ along with the one point functions $\langle\phi_I\rangle$ of the operators that appear will completely fix the two-point function via

$$\langle\phi_1(x_1)\phi_2(x_2)\rangle = \sum_{\phi_I} c_{\Delta,I}C_{\Delta,I}(x_{12},\partial_2)\langle\phi_I(x_2)\rangle \ . \tag{5.30}$$

As we learned above, there are severe restrictions on which $\langle\phi_I(x_2)\rangle$ are nonzero in the presence of a defect or boundary. In the boundary case, only scalars can have an expectation value. In higher codimension, operators in symmetric traceless representations of the Lorentz group with an odd number of indices, for example vectors, will have vanishing expectation value.

For simplicity, let us focus on the boundary case and the two-point function of two-scalars. In this case, we can be a little more specific about the precise form of the two-point function

$$\langle \phi_1(x_1)\phi_2(x_2)\rangle = \sum_{\Delta} c_\Delta C_\Delta(x_{12}, \partial_2)\frac{a_\Delta}{r_2^\Delta} = \sum_\Delta c_\Delta a_\Delta \frac{G_\Delta^{\rm bulk}(\xi_1)}{r_1^{\Delta_1} r_2^{\Delta_2}} \ , \tag{5.31}$$

where we have introduced a bulk conformal block $G_\Delta^{\rm bulk}(\xi)$. Because of our assumptions, $G_\Delta^{\rm bulk}(\xi)$ is an eigenfunction of the the full $SO(d,2)$ Casimir operator. (Note in the boundary case, the second cross ratio $\xi_2 = 1$ drops out.) A critical difference here from the four-point function case we considered above is the appearance of $c_\Delta a_\Delta$ in place of the three point coefficient squared $(c_\Delta)^2$. As we will see later in the context of the bootstrap, one is positive definite while the other is not.

We will not derive the bulk conformal blocks here, but for a bondary conformal field theory and two identical scalar operators, they take a form that could perhaps be guessed from the case without boundary. Consider the cross ratios u and v discussed above, or equivalently z and $\bar z$, constructed from the four insertion locations x_1, x_2, x_3 and x_4. If we treat x_3 as the image of x_1 and x_4 as the image of x_2, then z and $\bar z$ are degenerate and become equal to ξ_1. The bulk conformal block is proportional to the $k_\Delta(z)$ function defined above:

$$G_{\hat\Delta}^{\rm bulk}(\xi_1) = \xi_1^{-\Delta + \frac{\hat\Delta}{2}} {}_2F_1\left(\frac{\hat\Delta}{2}, \frac{\hat\Delta}{2}, 1 - \frac{d}{2} + \hat\Delta, -\xi_1 \right) \ . \tag{5.32}$$

There is a second type of OPE we can consider in the presence of a boundary or defect, where a local operator $\phi(x)$ gets close to the defect. In this case, we expect to be able to reconstruct the bulk operator from a sum over defect operators via the defect (or boundary) OPE:

$$\phi(x) = \sum_{\hat\phi_I} c_{\hat\Delta, I} \hat C_{\hat\Delta, I}(r, \partial_{\mathbf y}) \hat\phi_I(\mathbf y)\Big|_{\mathbf{y}=\mathbf{x}} \ . \tag{5.33}$$

Using a pair of these defect OPEs, we can reconstruct the two point function in a different way, via

$$\langle \phi(x_1)\phi(x_2)\rangle = \sum_{\hat\Delta, I} c_{\hat\Delta,I}^2 [C_{\hat\Delta,I}(r_1,\partial_{\mathbf y}) C_{\hat\Delta, I}(r_2, \partial_{\mathbf z}) \langle \hat\phi_{\hat\Delta,I}(\mathbf y)\hat\phi_{\hat\Delta,I}(\mathbf z)\rangle]\Big|_{\mathbf{y}=\mathbf{x_1}, \mathbf{z}=\mathbf{x_2}} \tag{5.34}$$

$$= \sum_{\hat\Delta, I} c_{\hat\Delta, I}^2 \frac{G_{\hat\Delta,I}^{\rm def}(\xi_1,\xi_2)}{r_1^{\Delta_1} r_2^{\Delta_2}} \tag{5.35}$$

where now we have the defect conformal blocks $G_{\hat\Delta,I}^{\rm def}(\xi_1,\xi_2)$. Note that the defect conformal blocks must be eigenfunctions of the Casimir operator for the reduced symmetry group $SO(p,2)\times SO(q)$.

A nice aspect of the $\langle\phi(x_1)\phi(x_2)\rangle$ two-point function in the presence of the boundary is that it is possible to construct the blocks $G^{\rm bry}_{\hat\Delta}(\xi_1)$ through pedestrian means. We start by deducing the form of $C_{\hat\Delta}(r,\partial)$ through comparison with the $\langle\phi(x)\hat\phi_\Delta(\mathbf{y})\rangle$ bulk-boundary two-point function:

$$\frac{c_{\hat\Delta}}{r^{\Delta-\hat\Delta}|\mathbf{x}^2+r^2|^{2\hat\Delta}} = \langle\phi(x)\hat\phi(0)\rangle = c_{\hat\Delta}C_{\hat\Delta}(r,\partial_{\mathbf{x}})\langle\hat\phi_{\hat\Delta}(\mathbf{x})\hat\phi_{\hat\Delta}(0)\rangle = c_{\hat\Delta}C_{\hat\Delta}(r,\partial_{\mathbf{x}})\frac{1}{|\mathbf{x}|^{2\hat\Delta}} \qquad (5.36)$$

Next, we expand out the left hand side in a Taylor series near the boundary:

$$\begin{aligned}\frac{1}{r^{\Delta-\hat\Delta}|\mathbf{x}^2+r^2|^{\hat\Delta}} &= \frac{1}{|r|^{\Delta-\hat\Delta}|\mathbf{x}|^{2\hat\Delta}\left|1+\frac{r^2}{\mathbf{x}^2}\right|^{\hat\Delta}} \\ &= \frac{1}{r^{\Delta-\hat\Delta}|\mathbf{x}|^{2\hat\Delta}}\sum_{j=0}^\infty \frac{(\hat\Delta)_j}{j!}\left(-\frac{r^2}{|\mathbf{x}|^2}\right)^j .\end{aligned} \qquad (5.37)$$

We have introduced the Pochammer symbol $(a)_j = a(a+1)\cdots(a+j-1)$. As a next step, we want to express the sum as a derivative operator acting on $|\mathbf{x}|^{-2\hat\Delta}$. To do that, we need to know how the $(d-1)$-dimensional boundary Laplacian acts on powers of $|\mathbf{x}|$. In particular, we have

$$\Box\frac{1}{|\mathbf{x}|^\beta} = \frac{\beta(\beta+3-d)}{|\mathbf{x}|^{\beta+2}} , \qquad (5.38)$$

and hence

$$\begin{aligned}\Box^j\frac{1}{|\mathbf{x}|^{2\hat\Delta}} &= \frac{(2\hat\Delta)(2\hat\Delta+2)\cdots(2\hat\Delta+2j-2)(2\hat\Delta-d+3)(2\hat\Delta-d+5)\cdots(2\hat\Delta-d+2j-1)}{|\mathbf{x}|^{2\hat\Delta+2j}} \\ &= \frac{2^{2j}(\hat\Delta)_j\left(\hat\Delta+\frac{3-d}{2}\right)_j}{|\mathbf{x}|^{2\hat\Delta+2j}}\end{aligned} \qquad (5.39)$$

Inserting this result into our expansion of the bulk-defect two-point function yields

$$\frac{1}{r^{\Delta-\hat\Delta}|\mathbf{x}^2+r^2|^{\hat\Delta}} = \frac{1}{r^{\Delta-\hat\Delta}}\sum_{j=0}^\infty\frac{(-r^2)^j}{j!2^{2j}\left(\hat\Delta+\frac{3-d}{2}\right)_j}\Box^j\frac{1}{|\mathbf{x}|^{2\hat\Delta}} . \qquad (5.40)$$

which allows us to read off a representation of $C_{\hat\Delta}(r,\partial)$.

The next step is to construct the conformal block

$$\begin{aligned}G^{\rm bry}_{\hat\Delta}(\xi_1) &= r_1^{\Delta_1}r_2^{\Delta_2}C_{\hat\Delta}(r_1,\partial_1)C_{\hat\Delta}(r_2,\partial_2)\frac{1}{|\mathbf{x}_1-\mathbf{x}_2|^{2\hat\Delta}} \qquad (5.41)\\ &= (r_1r_2)^{\hat\Delta}\sum_{j,k}\frac{(-r_1^2)^j(-r_2^2)^k}{j!k!2^{2(j+k)}\left(\hat\Delta+\frac{3-d}{2}\right)_j\left(\hat\Delta+\frac{3-d}{2}\right)_k}\Box_1^j\Box_2^k\frac{1}{|\mathbf{x}_1-\mathbf{x}_2|^{2\hat\Delta}} \\ &= (r_1r_2)^{\hat\Delta}\sum_{j,k}\frac{(-r_1^2)^j(-r_2^2)^k(\hat\Delta)_{j+k}\left(\hat\Delta+\frac{3-d}{2}\right)_{j+k}}{j!k!\left(\hat\Delta+\frac{3-d}{2}\right)_j\left(\hat\Delta+\frac{3-d}{2}\right)_k}\frac{1}{|\mathbf{x}_1-\mathbf{x}_2|^{2\hat\Delta+2j+2k}} . \qquad (5.42)\end{aligned}$$

At this point, we invoke the following result[8] for a double sum

$$\sum_{j,k=0}^{\infty} \frac{1}{j!k!} \frac{(\lambda)_{j+k}(\kappa)_{j+k}}{(\kappa)_j(\kappa)_k} \frac{A^j B^k}{z^{2(\lambda+j+k)}} = \frac{1}{C^\lambda} \sum_{j=0}^{\infty} \frac{(\lambda)_{2j}}{j!(\kappa)_j} \left(\frac{AB}{C^2}\right)^j , \tag{5.43}$$

where $z^2 = A + B + C$. Note that

$$\frac{AB}{C^2} = \frac{r_1^2 r_2^2}{(|\mathbf{x}_1 - \mathbf{x}_2|^2 + r_1^2 + r_2^2)^2} = \frac{1}{4(1+2\xi_1)^2} \tag{5.44}$$

This sum allows us to write the conformal block in the form

$$G^{\text{bry}}_{\hat{\Delta}}(\xi_1) \quad = \quad (2(1+2\xi_1))^{-\hat{\Delta}} \sum_{j=0}^{\infty} \frac{(\hat{\Delta})_{2j}}{j! \left(\hat{\Delta} + \frac{3-d}{2}\right)_j} \left(\frac{1}{4(1+2\xi_1)^2}\right)^j \tag{5.45}$$

which is almost the series definition of a hypergeometric function

$$G^{\text{bry}}_{\hat{\Delta}}(\xi_1) = (2(1+2\xi_1))^{-\hat{\Delta}} {}_2F_1\left(\frac{\hat{\Delta}}{2}, \frac{\hat{\Delta}+1}{2}, \hat{\Delta} + \frac{3-d}{2}, \frac{1}{(1+2\xi_1)^2}\right) \tag{5.46}$$

There is a hypergeometric identity[9] which allows us to give a slightly simpler representation of the conformal block in the ξ_1 variable:

$$G^{\text{bry}}_{\hat{\Delta}}(\xi_1) = \xi_1^{-\hat{\Delta}} {}_2F_1\left(\hat{\Delta}, 1 - \frac{d}{2} + \hat{\Delta}, 2 - d + 2\hat{\Delta}, -\frac{1}{\xi_1}\right) . \tag{5.47}$$

6 The Conformal Bootstrap

Say we just took out of a hat a random set of conformal primaries and OPE coefficients. Would such a selection provide the data to define a CFT? We would quickly find that such a random selection would lead to an inconsistent procedure for generating four and higher point correlation functions.

Consider the correlation function of four identical scalars of dimension η:

$$\langle \Phi(x_1)\Phi(x_2)\Phi(x_3)\Phi(x_4) \rangle . \tag{6.1}$$

At the end of the previous section, we decomposed this object into a sum over conformal blocks by taking x_1 close to x_2 and x_3 close to x_4. However, we could equally well have proceeded by taking instead x_1 close to x_4 and x_2 close to x_3. This alternate procedure is

[8]This result is mentioned in (B.8) of F. A. Dolan and H. Osborn, "Conformal four point functions and the operator product expansion," Nucl. Phys. B **599**, 459-496 (2001) [arXiv:hep-th/0011040 [hep-th]].

[9]Note this result does not give the correct answer for the sum in the special case $\hat{\Delta} = \frac{d-2}{2}$. In this case, (5.45) and (5.46) evaluate to

$$\frac{1}{2^{d-1}} \left(\xi_1^{-\hat{\Delta}} + (1+\xi_1)^{-\hat{\Delta}}\right) .$$

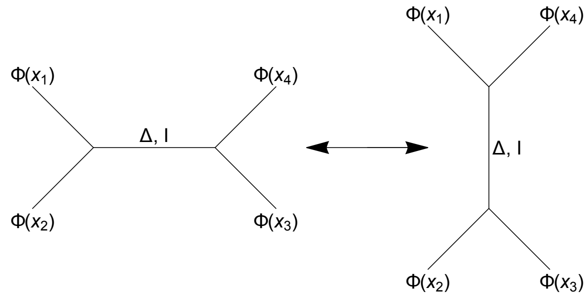

Figure 4: The basic crossing symmetry constraint.

equivalent to swapping x_2 and x_4 in the original decomposition. From the form of the cross ratios

$$u = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2} \ , \quad v = \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2} \ ,$$

this swap also exchanges u and v. We learn that

$$\frac{G(u,v)}{|x_{12}|^{2\eta}|x_{34}|^{2\eta}} = \frac{G(v,u)}{|x_{14}|^{2\eta}|x_{23}|^{2\eta}} \tag{6.2}$$

or equivalently $v^\eta G(u,v) = u^\eta G(v,u)$. Inserting the partial wave decomposition, this relation becomes

$$v^\eta \sum_{\Delta,I} c_{\Delta,I}^2 G_{\Delta,I}(u,v) = u^\eta \sum_{\Delta,I} c_{\Delta,I}^2 G_{\Delta,I}(v,u) \ . \tag{6.3}$$

The exchange is illustrated pictorially in figure 4.

Now there is one operator in the spectrum of every CFT on whose presence we can rely, the identity operator. This operator has no descendants because the momentum operator annihilates constant valued functions. The OPE coefficient of $\phi \times \phi$ with the identity can be taken to be one, assuming we have normalized our two point functions conventionally, to have the form $|x-y|^{-2\eta}$. Removing the identity operator from the partial wave decomposition, we find

$$v^\eta \left(1 + {\sum_{\Delta,I}}' c_{\Delta,I}^2 G_{\Delta,I}(u,v)\right) = u^\eta \left(1 + {\sum_{\Delta,I}}' c_{\Delta,I}^2 G_{\Delta,I}(v,u)\right) \ . \tag{6.4}$$

The conformal bootstrap equation is then the following slight massage of the previous expression:

$${\sum_{\Delta,I}}' c_{\Delta,I}^2 \left(\frac{v^\eta G_{\Delta,I}(u,v) - u^\eta G_{\Delta,I}(v,u)}{u^\eta - v^\eta}\right) = 1 \ . \tag{6.5}$$

Generically, a random selection of conformal primaries and their OPE coefficients will be inconsistent with this relation. One could take a step back and insist only on a random selection of conformal primaries. Perhaps then the $c_{\Delta,I}$ can be adjusted to make the equation

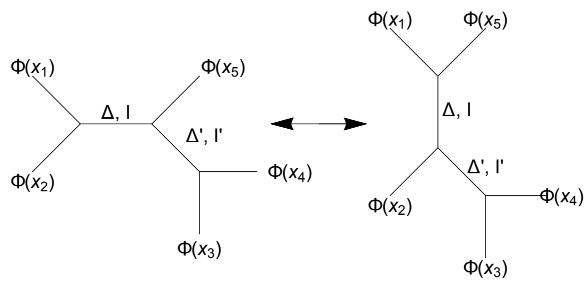

Figure 5: A crossing symmetry constraint for a five-point function.

true. In fact, however, one can use this expression to place bounds on the operator spectrum as well!

Before we proceed further with trying to constrain the operator spectrum, a natural question to ask is whether considering higher point functions will lead to further constraints on the set of possible conformal field theories. The answer is no. By imposing four point crossing symmetry on intermediate channels of higher point functions, one can access all possible ways of decomposing the higher point functions into conformal blocks. The case of a five point function is illustrated in fig. 5. From a more formal standpoint, we are making a statement about the associativity of the operator algebra.

6.1 Interlude on Unitarity Bounds

In order to determine these bounds on the operator spectrum, one imposes additionally unitarity. Unitarity implies that the dimension of a field of spin ℓ in a symmetric traceless representation is bounded below by

$$\begin{aligned}
\Delta &\geq \ell + d - 2 \text{ if } \ell = 1, 2, 3, \ldots \\
\Delta &\geq \frac{d-1}{2} \text{ if } \ell = \frac{1}{2} \, , \\
\Delta &\geq \frac{d-2}{2} \text{ if } \ell = 0 \, .
\end{aligned}$$

It also imposes that the OPE coefficients are real, so that $c^2_{\Delta,I} \geq 0$. Note that the minimum dimension of a scalar $\frac{d-2}{2}$ is the engineering dimension of a free scalar in d dimensions. The minimum dimension for $\ell = \frac{1}{2}$ is the engineering dimension of a free spin one half fermion. The minimum dimension for a vector field $\ell = 1$ is in fact the dimension of a conserved current. Similarly, the minimum dimension of a symmetric, traceless spin two field is the same as the dimension of the stress tensor. In other words, the dimensions of a generic field in CFT must be, according to its spin, greater or equal to that of a free scalar, free fermion, conserved current or stress tensor. There is a pattern here, that the multiplets generated from a primary of the smallest conformal dimension tend to be smaller. There is a shortening condition, where some of the descendants vanish. In the case of the free scalar, the condition is that $\Box\phi = 0$. For the fermion, it's the Dirac equation. For the conserved current and stress tensor, that $\partial_\mu J^\mu = 0 = \partial_\mu T^{\mu\nu}$.

Let us try to understand where these bounds come from in more detail. When we talk about unitarity for a Euclidean CFT (where time is like all the other spatial coordinates), what we really mean is reflection positivity:

$$\langle \mathcal{R}(\mathcal{O})\mathcal{O}\rangle \geq 0 \ , \tag{6.6}$$

where $\mathcal{O}$ is some arbitrary operator, possibly composite, and $\mathcal{R}$ is a reflection operator that reflects all of the insertions in $\mathcal{O}$ about some plane, for example $\{(x_1, x_2, \ldots) : x_1 = 0\}$ or $\{(x_1, x_2, \ldots) : x_1 = 1\}$. Indeed, since this is a conformal field theory, we can act on the space with special conformal transformations which will in general turn planes into spheres.

Problem 6.1. *Show that a special conformal transformation with $b^\mu = (1, \vec{0})$ turns the plane $x^\mu = (1/2, \vec{0})$ into a sphere centered at the origin of radius one. Furthermore, show that reflection about the plane $x_1 = 1/2$ becomes inversion after the special conformal transformation.*

Thus another way of insisting on reflection positivity is to work with the cylindrical coordinate system from section 5 where $\tau = \log r$ and to claim

$$\langle \mathcal{T}(\mathcal{O})\mathcal{O}\rangle \geq 0 \ . \tag{6.7}$$

where $\mathcal{T}$ sends $\tau \to -\tau$ (or equivalently $r \to 1/r$) in all the insertions that make up $\mathcal{O}$. In a Lorentzian context, Wick rotating time $\tau \to it$, we can then sometimes go further and think of $\mathcal{T}(\mathcal{O})$ as a Hermitian conjugate $\mathcal{O}^\dagger$.

From our experience building up representations of the conformal algebra using P_μ and K_μ, we saw that P_μ functioned like a raising operator while K_μ was a lowering operator. Given this intuition, let us see whether there is some sense in which K_μ can be treated as a reflection (or Hermitian conjugate) of P_μ. We make the change of variables $x_\mu = e^\tau n_\mu$ and hence $\tau = \frac{1}{2}\log x^2$ and $n_\mu = x_\mu/\sqrt{x^\nu x_\nu}$. We find then that

$$\begin{aligned} iP_\mu &= \partial_\mu = \left(\frac{\partial\tau}{\partial x^\mu}\frac{\partial}{\partial\tau} + \frac{\partial n_\nu}{\partial x^\mu}\frac{\partial}{\partial n_\nu}\right) \\ &= e^{-\tau}\left(n_\mu\frac{\partial}{\partial\tau} + (\delta_{\mu\nu} - n_\mu n_\nu)\frac{\partial}{\partial n_\nu}\right) \ . \end{aligned} \tag{6.8}$$

Building off the previous result, we quickly see that for the special conformal transformation

$$iK_\mu = \left[x^2\partial_\mu - 2x_\mu(x\cdot\partial)\right] = e^\tau\left[-n_\mu\frac{\partial}{\partial\tau} + (\delta_{\mu\nu} - n_\mu n_\nu)\frac{\partial}{\partial n_\nu}\right] \ . \tag{6.9}$$

In other words $\mathcal{T}(iP_\mu) = iK_\mu$. We have swept a factor of -1 under the rug here by including some extra factors of i. This factor deserves a longer discussion that I would prefer not to get into here.

Consider now a primary state $|\phi_I\rangle$, pushing our insertions off to $\tau \to \pm\infty$. From reflection positivity follow a number of claims, two of which will be important for us:

$$\begin{aligned} -\langle\phi_I|K_\mu P_\nu|\phi_J\rangle &\geq 0 \ , & (6.10) \\ \langle\phi_I|K_\mu K_\nu P_\lambda P_\rho|\phi_J\rangle &\geq 0 \ , & (6.11) \end{aligned}$$

are non-negative as matrices (i.e. all the eigenvalues are zero or positive). Applying the commutation relation of translations with special conformal transformations to (6.10), along with the constraint that K_μ annihilates $|\phi_J\rangle$, we find that

$$-2i\langle\phi_I|(\delta_{\mu\nu}D - M_{\mu\nu})|\phi_J\rangle \geq 0 \ . \tag{6.12}$$

(Notice we have replaced $\eta_{\mu\nu}$ with $\delta_{\mu\nu}$ because we are working with a Euclidean theory, not a Lorentzian one.) For a scalar operator, $M_{\mu\nu}$ will annihilate $|\phi_J\rangle$, and we find the constraint that $\Delta \geq 0$. Comparing with (6.6), you may be confused because the scalar is supposed to be bounded below by $\frac{d-2}{2}$ while we just found the constraint $\Delta > 0$. In fact, $\Delta = 0$ must be allowed, as it corresponds to the identity operator. What happens more precisely is that there is a gap in the spectrum and the next allowable dimension is that of a free field, $\frac{d-2}{2}$. To see this, one has to consider (6.11):

Problem 6.2. *By studying $\langle\phi|K^2P^2|\phi\rangle$ for scalar primary ϕ, demonstrate that the conformal dimension must satisfy the quadratic constraint $\Delta(2(\Delta+1)-d) \geq 0$.*

A little bit of group theory allows one to analyze the general case of (6.12), which we will not do here. However, we know how to represent $M_{\mu\nu}$ for spinors and vectors from chapter 2, from which you can deduce the corresponding bounds (6.6):

Problem 6.3. *Use the explicit representation of $M_{\mu\nu}$ from problem 2.2 for spinors and vectors to show that Δ is bounded below by $\frac{d-1}{2}$ and $d-1$ respectively.*

A natural question is if any further constraints on the spectrum can be found by considering more complicated correlation functions involving K_μ and P_μ. The answer appears to be no.

One way to argue that three point function coefficients are real in CFT is to consider $\langle\mathcal{R}(\mathcal{O})\mathcal{O}\rangle$ where $\mathcal{O} = \phi_1(x_1)\phi_2(x_2)\phi_3(x_3)$. By taking a reflection plane that is very far from the insertions, we expect the dominant contribution of this six point function to be of the form $\langle\mathcal{R}(\mathcal{O})\mathcal{O}\rangle \sim \langle\mathcal{R}(\mathcal{O})\rangle\langle\mathcal{O}\rangle$. For this quantity to be positive, the three point functions need to be real.

6.2 The Bootstrap

Now let us define

$$F_{\Delta,I}(u,v) \equiv \frac{v^\eta G_{\Delta,I}(u,v) - u^\eta G_{\Delta,I}(v,u)}{u^\eta - v^\eta} \ . \tag{6.13}$$

and imagine that we have found a candidate spectrum for the theory. We have some set, possibly infinite, of dimensions for scalar operators, some set of dimensions for vector operators, and so on. Now we design a linear operator $\mathcal{O}$ such that

$$\mathcal{O}(F_{\Delta,I}(u,v)) \ \ \geq \ \ 0 \ , \tag{6.14}$$

for every operator in the spectrum but $\mathcal{O}(1) < 0$. Then, because we know $c_{\Delta,I}^2 > 0$, we can rule this spectrum out as possible data for a CFT. In fact, by cleverly choosing $\mathcal{O}$, it is possible to rule out whole families of possible CFTs.

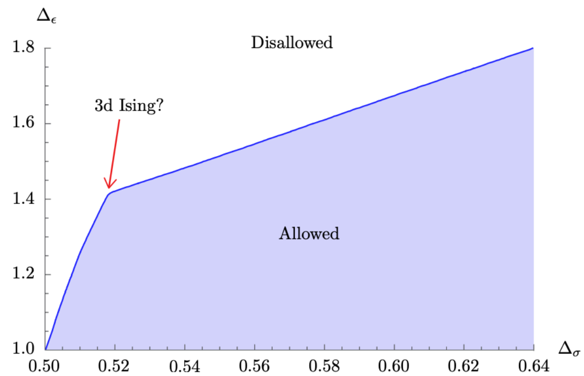

Figure 6: Upper bound on the dimension of Δ_ϵ of the lowest dimension scalar in the $\sigma \times \sigma$ OPE, where σ is a real scalar primary in a unitary 3d CFT with a $\mathbb{Z}_2$ symmetry. [[From Simmons-Duffin's TASI lectures]]

Let us consider a CFT with a scalar operator σ of dimension Δ_σ. The OPE of two such scalars will have the generic form

$$\sigma(x)\sigma(0) = \frac{1}{|x|^{2\Delta_\sigma}} \left(1 + C_{\sigma\sigma\epsilon}|x|^{\Delta_\epsilon}\epsilon(0) + \ldots\right) \ , \tag{6.15}$$

where $\epsilon(x)$ is the leading operator to appear in the OPE after the identity. In a free CFT, we anticipate that $\epsilon(x)$ will be the normal ordered product of $\sigma(x)$ with itself. In this case, $\Delta_\sigma = \frac{d-2}{2}$ and $\Delta_\epsilon = d-2$. But more generally, it is not obvious what Δ_ϵ should be. By applying the bootstrap procedure, we can determine an upper bound for Δ_ϵ as a function of Δ_σ. See fig. 6. Reassuringly, the point $(\frac{1}{2}, 1)$ lies on the bounding curve in $d=3$. Moreover, there is a kink in the bounding curve close to the location of the 3d Ising model.

In fact, by imposing crossing symmetry on more than one four point function, one can often further pin down the data of interesting CFTs. For example, the most accurate data for the 3d Ising model at the critical point currently come from bootstrap bounds:[10]

$$(\Delta_\sigma, \Delta_\epsilon) = (0.518151(6), 1.41264(6)) \ . \tag{6.16}$$

One might ask if these results have some experimental relevance. Recall the 3d Ising model has Hamiltonian

$$H = -J \sum_{\langle i,j \rangle} s_i \cdot s_j \ , \tag{6.17}$$

where $s_i = \pm 1$ and the sum is over nearest neighbors on a 3d cubic lattice. When we talk about the CFT associated with the Ising model, we mean the CFT that describes the behavior of the lattice model at the critical temperature, where it is on the border between an

[10]D. Simmons-Duffin, A Semidefinite Program Solver for the Conformal Bootstrap, JHEP 06 (2015) 174, arxiv.org/abs/1502.02033.

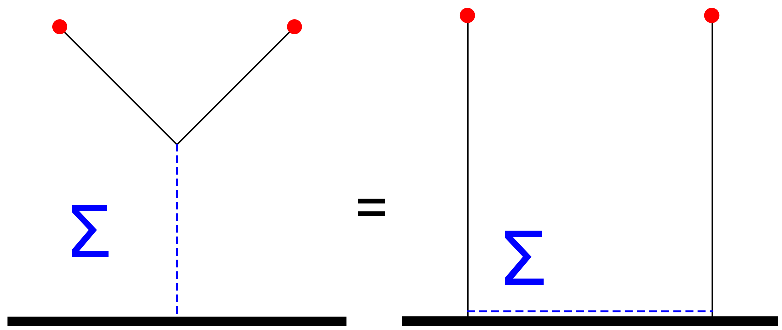

Figure 7: In the presence of a defect, there is a bulk-defect crossing symmetry constraint.

ordered low temperature and a disordered high temperature system. While I am not aware of a measurement of the critical exponents for Ising, there is one for a small generalization of Ising. We can promote $\vec{s}_i$ to n-component vectors of unit length. In the case $n = 2$, the associated CFT is believed to also describe helium along the line in the temperature-pressure plane that separates the superfluid from the ordinary fluid.

The analog of Δ_ϵ above for the $n = 2$ model was calculated from a bootstrap approach to be 1.51136(22).[11] However, the experiment (which needs to be done in space to avoid the effects of gravity) measured 1.50946(22). This discrepancy is eight standard deviations, which as far as I am aware, remains unexplained. My reading is that while it seems likely that the theoretical result is correct is far as it goes, the physics measured by the experiment may not be precisely that of a CFT. The situation is unsatisfactory, and I leave it to one of you to improve the story in the next retelling.

6.3 The Boundary Bootstrap

There is a similar crossing symmetry constraint for two-point functions in boundary and defect CFT. In particular, the sum over bulk conformal blocks from the bulk OPE of the two operators must agree with the sum over the boundary conformal blocks from the defect OPE (see figure 7).

It is illuminating to consider a very simple example of crossing symmetry in a boundary CFT. Consider the free scalar field that we introduced before. In the presence of a boundary, we expect the scalar field will satisfy either Dirichlet $\phi = 0$ or Neumann $\partial_r \phi = 0$ boundary conditions, leading to two possible two-point functions:

$$\langle \phi(x_1)\phi(x_2)\rangle = \frac{1}{(|\mathbf{x}_1 - \mathbf{x}_2|^2 + (r_1 - r_2)^2)^{\Delta_\phi}} + \frac{\chi}{(|\mathbf{x}_1 - \mathbf{x}_2|^2 + (r_1 + r_2)^2)^{\Delta_\phi}} , \qquad (6.18)$$

where $\chi = 1$ for Neumann and $\chi = -1$ for Dirichlet. To avoid mess, let $\Delta_\phi = \frac{d-2}{2}$ be the scaling dimension of the free field ϕ. We have chosen the conventional unit normalization, fixed by the behavior of the two-point function in the limit $x \to x'$ where the effect of the

[11] Chester et al., Carving out OPE space and precise O(2) critical exponents, JHEP 06 (2020) 142, arxiv.org/abs/1912.03324.

boundary can be ignored. We next re-express this two point function using the cross ratio ξ_1, putting it in the form (3.66):

$$\langle\phi(x_1)\phi(x_2)\rangle = \frac{G(\xi_1)}{(r_1 r_2)^{\Delta_\phi}} = \frac{1}{(r_1 r_2)^{\Delta_\phi}}\frac{1}{2^{d-2}}\left(\xi_1^{-\Delta_\phi} + \chi(1+\xi_1)^{-\Delta_\phi}\right) . \tag{6.19}$$

Let us decompose $G(\xi_1)$ into bulk and boundary conformal blocks. First, we note that

$$G^{\rm bry}_{\Delta_\phi} = 2\left(\xi_1^{-\Delta_\phi} + (1+\xi_1)^{-\Delta_\phi}\right) , \tag{6.20}$$

$$G^{\rm bry}_{\Delta_\phi+1} = \frac{2}{d-2}\left(\xi_1^{-\Delta_\phi} - (1+\xi_1)^{-\Delta_\phi}\right) . \tag{6.21}$$

The choice of scaling dimension is motivated by the fact that there should be a boundary operator of dimension $\hat{\Delta} = \frac{d-2}{2}$ corresponding to the boundary limit of ϕ and a second of dimension $\hat{\Delta} = \frac{d}{2}$ corresponding to the boundary limit of $\partial_r\phi$. In the cases of Dirichlet or Neumann boundary conditions, one or the other of these operators should be absent.

At the same time, we have the two bulk blocks (5.32):

$$G^{\rm bulk}_0 = \xi_1^{-\Delta_\phi} , \tag{6.22}$$

$$G^{\rm bulk}_{2\Delta_\phi} = (1+\xi_1)^{-\Delta_\phi} . \tag{6.23}$$

The choice of scaling dimension here is motivated by the fact that ϕ has a nonzero two-point function with itself and so the bulk identity operator should be present in its OPE. Also, we expect the (normal ordered) operator ϕ^2 to be present as well. Apparently, in this simple case, no other blocks are needed. We can write the following solution of the crossing symmetry constraint for general χ:

$$G^{\rm bulk}_0 + \chi G^{\rm bulk}_{2\Delta_\phi} = \frac{1+\chi}{2}\frac{1}{2}G^{\rm bry}_{\Delta_\phi} + \frac{1-\chi}{2}\frac{d-2}{2}G^{\rm bry}_{\Delta_\phi+1} . \tag{6.24}$$

It turns out more general values of χ (more general than ± 1) can be accessed by coupling the bulk ϕ field to boundary degrees of freedom, for example a minimal model on the boundary.[12] This case is deceptively simple. Generically, these sums over conformal blocks involve infinite towers of operators, with intricate convergence properties.

Problem 6.4. *Show that reflection positivity for $\langle\phi\,\phi\rangle$ and $\langle\partial_r\phi\,\partial_r\phi\rangle$ constrain χ to lie in the range $-1 \leq \chi \leq 1$.*

As discussed above, the coefficients of the boundary blocks must be positive. However, the coefficients of the bulk blocks can be of either sign, making standard implementations of the bootstrap program more difficult.

[12] C. Behan, L. Di Pietro, E. Lauria and B. C. van Rees, "Bootstrapping boundary-localized interactions II: Minimal models at the boundary," [arXiv:2111.04747 [hep-th]].

6.4 ϕ^4 Theory

Consider the action:

$$S = \int \mathrm{d}^d x \left[\frac{1}{2}(\partial\phi)^2 + \frac{g}{4!}\phi^4 \right] . \tag{6.25}$$

The beta function for the coupling in this theory in $d = 4 - \epsilon$ dimensions is

$$\beta = -\epsilon g + \frac{3}{16\pi^2} g^2 + O(g^3) , \tag{6.26}$$

which has a nontrivial zero at

$$g_* = \frac{16\pi^2}{3}\epsilon + O(\epsilon^2) . \tag{6.27}$$

Let us see if we can find anomalous dimensions for some of the operators in this theory using our knowledge of CFT without working too hard.

The first observation is almost a cheat, namely that for the ϕ^4 operator, with $\Delta_{\phi^4} = 2d - 4 + \gamma_{\phi^4}$, $\gamma_{\phi^4} = 2\epsilon + O(\epsilon^2)$. This observation follows directly from the beta function – that the classical dimension of ϕ^4, $4 - 2\epsilon$, has to cancel against the anomalous dimension to guarantee that g is exactly marginal. In fact, the beta function also involves the wave function renormalization of the kinetic term, or equivalently the anomalous dimension of ϕ itself. However, in ϕ^4 theory, this anomalous dimension is $O(\epsilon^2)$, as we will see below.

Now if we know ϕ^4 has anomalous dimension 2ϵ, it follows trivially that for $n > 2$

$$\gamma_{\phi^n} = \frac{n(n-1)}{6}\epsilon + O(\epsilon^2) . \tag{6.28}$$

The reason is that the anomalous dimension will come from a family of Feynman diagrams that compute $\langle \phi^n \phi^n \rangle$ at leading order in g where we contract two legs from one ϕ^n with two legs from another and a ϕ^4 vertex. There are n choose two ways of drawing these diagrams, all of which give the same anomalous contribution to the dimension. For $n = 1$, of course, there is no such diagram, and the leading order correction to the anomalous dimension will vanish.

Let's work a little harder and try to compute γ_ϕ. The equation of motion which follows from the action is

$$\Box\phi = \frac{g}{6}\phi^3 . \tag{6.29}$$

We impose this equation of motion on the two-point function

$$\langle \phi(x)\phi(0) \rangle = \frac{\kappa}{|x|^{2\Delta}} . \tag{6.30}$$

(With the canonically normalized kinetic term in the action, we should set $\kappa^{-1} = (d - 2)\,\mathrm{Vol}(S^{d-1})$. In $d = 4$, this reduces to $\kappa^{-1} = 4\pi^2$.) As the coupling may affect the scaling dimension of ϕ, for the moment we are agnostic about the value of Δ. We determined (in the discussion of boundary conformal blocks) that the d-dimensional Laplacian acting on a power-law (5.38) gives the following result

$$\langle \Box\phi(x)\Box\phi(0)\rangle = \kappa\, 2\Delta(2\Delta+2)(2\Delta+2-d)(2\Delta+4-d)\frac{1}{|x|^{2\Delta+4}} \tag{6.31}$$

which vanishes, as it should, when Δ takes its free field value. Because of this small prefactor $(2\Delta+2-d)$, we are free to set other appearances of Δ to their free field value of 1:

$$\langle \Box\phi(x)\Box\phi(0)\rangle \approx 32\kappa\gamma_\phi \frac{1}{|x|^6} \ , \tag{6.32}$$

where we have defined the anomalous dimension of ϕ, $\gamma_\phi \equiv \Delta - \frac{d-2}{2}$.

Now ϕ^3 by the equations of motion becomes a level two descendant of ϕ once g is turned on, $\Box\phi \sim \phi^3$. Hence it's conformal dimension should be $\Delta+2$. However, in the limit $g=0$, we may consider it to be its own primary. Its two point function takes the form

$$\langle \phi^3(x)\phi^3(0)\rangle = \frac{6\kappa^3}{|x|^{2\Delta+4}} \ . \tag{6.33}$$

Inside the equation of motion, this expression gets multiplied by the small parameter g^2 and we thus are free to set $\Delta = 1$. Applying the equation of motion, we learn

$$\left(\frac{g}{6}\right)^2 6\kappa^3 = 32\kappa\gamma_\phi \tag{6.34}$$

allowing us to fix the anomalous dimension of ϕ to be

$$\gamma_\phi = \frac{6\kappa^2}{32}\frac{g^2}{36} = \frac{\epsilon^2}{108} \ , \tag{6.35}$$

without having had to compute a single loop diagram!

Let us move on to the case with a boundary. We saw before how to solve the crossing constraints in the free case. Now let us turn on a small g and try to solve the crossing constraints at linear order in g. We claim that there remains only one boundary field at play, either the boundary value of ϕ or of $\partial_n\phi$ depending on the boundary conditions we apply. In the bulk OPE, however, there is a small change with $g \neq 0$. Namely, we can find ϕ^4 in the bulk OPE of two scalars. Any higher powers than ϕ^4 will require more bulk vertices and hence higher powers of g, which we do not need to consider at this order. Thus, for Neumann boundary conditions, we should try to solve the crossing equation

$$G_0^{\rm bulk} + g c_{\phi^2} G^{\rm bulk}_{\Delta_{\phi^2}} + g c_{\phi^4} G^{\rm bulk}_{\Delta_{\phi^4}} = \mu^2 G^{\rm bry}_{\Delta_{\hat\phi}} \tag{6.36}$$

where our ansatz for the solution takes into account the earlier solution in the free limit $g = \epsilon = 0$:

$$\begin{aligned}
\Delta_\phi &= \frac{d-2}{2} + \gamma_\phi \ , \quad \Delta_{\phi^2} = d-2+\gamma_{\phi^2} \ , \quad \Delta_{\phi^4} = 2(d-2)+\gamma_{\phi^4} \ , \\
g c_\phi^2 &= 1 + \delta(g c_{\phi^2})\epsilon + O(\epsilon^2) \ , \quad g c_\phi^4 = \delta(g c_{\phi^4})\epsilon + O(\epsilon^2) \ , \\
\Delta_{\hat\phi} &= \frac{d-2}{2} + \gamma_{\hat\phi} \ , \quad \mu^2 = \frac{1}{2} + \delta\mu^2\,\epsilon + O(\epsilon^2) \ .
\end{aligned}$$

Note that since gc_{ϕ^4} is already $O(\epsilon)$, the anomalous dimension γ_{ϕ^4} should drop out of the equations. Expanding out the conformal blocks close to $d = 4$ and the free field case, we find

$$\gamma_\phi = O(\epsilon^2) \ , \quad \gamma_{\phi^2} = 2\alpha\epsilon + O(\epsilon^2) \ , \quad \gamma_{\hat{\phi}} = -\alpha\epsilon + O(\epsilon^2) \ , \tag{6.37}$$

$$\delta g(c_{\phi^2}) = \alpha \ , \quad \delta(gc_{\phi^4}) = \frac{\alpha}{2} \ , \quad \delta\mu^2 = 0 \ . \tag{6.38}$$

We computed above that $\gamma_\phi = \frac{\epsilon^2}{108}$, and so it is no surprise that we find no linear order contribution. Also it is curious but characteristic of these models that the boundary operator $\hat{\phi}$, which in the free limit is just the boundary limit of ϕ, picks up a different anomalous dimension than the bulk ϕ. To fix α, we need some external input, for example the anomalous dimension $\gamma_{\phi^2} = \frac{\epsilon}{3} + O(\epsilon^2)$. One can go through a similar exercise for the Dirichlet boundary condition, replacing $G^{\text{bry}}_{\Delta_{\hat{\phi}}}$ on the right hand side with $G^{\text{bry}}_{\Delta_{\partial_n\hat{\phi}}}$. The results for the anomalous dimensions are the same in both cases. Note this same class of solutions will hold in general for the $O(N)$ model. All we have to do is replace the anomalous dimension for ϕ^2 with the general result $\gamma_{\phi^2} = \frac{N+2}{N+8}\epsilon + O(\epsilon^2)$.

This ϕ^4 model and generalization to the $O(N)$ model, ϕ_i, $i = 1, \ldots, N$, are argued to describe second order phase transitions in systems with boundary. There are even some experimental measurements of boundary critical exponents that compare favorably to results from the ϵ-expansion and variants of the bootstrap.

Let us try to understand the phase diagram of these models. There is an additional relevant operator we can add to the $O(N)$ model with boundary,

$$S_{\text{bry}} = \frac{m_{\text{bry}}}{2} \int_{x^n=0} \mathrm{d}^{d-1}x \, \phi^i \phi_i \ , \tag{6.39}$$

additional to the bulk mass term $\int_{x^n>0} \mathrm{d}^d x \, m^2 \phi^i \phi_i$. From a variational principle, such a term leads to the following classical boundary condition for the scalar field,

$$\partial_n \phi_i = m_{\text{bry}} \phi_i \ . \tag{6.40}$$

First tune the bulk m to the critical point where the bulk begins to order. We can further tune to a surface critical point $m_{\text{bry}} \approx 0$ (we say approximately, because there are quantum corrections here that spoil the classical analysis) where we have Neumann boundary conditions. For $m_{\text{bry}} > 0$, at low energies E we expect this parameter m/E becomes effectively infinite, forcing Dirichlet boundary conditions. Interestingly, there is a third possibility, $\phi_i \sim \frac{1}{r^{\Delta_\phi}}$, which can occur when $m_{\text{bry}} < 0$. In this case, the surface has ordered before the bulk and one has a further extraordindary transition where the bulk begins to order in the presence of surface order. A 2d phase diagram emerges from the different possibilities (see fig. 8).

Let's try to compare some of these anomalous dimensions to experiment. Let's focus on the "ordinary" or Dirichlet boundary condition case. We want to look at the surface magnetization of a metal, which should obey the scaling law

$$\text{magnetization} \sim |T_c - T|^{\hat{\beta}} \tag{6.41}$$

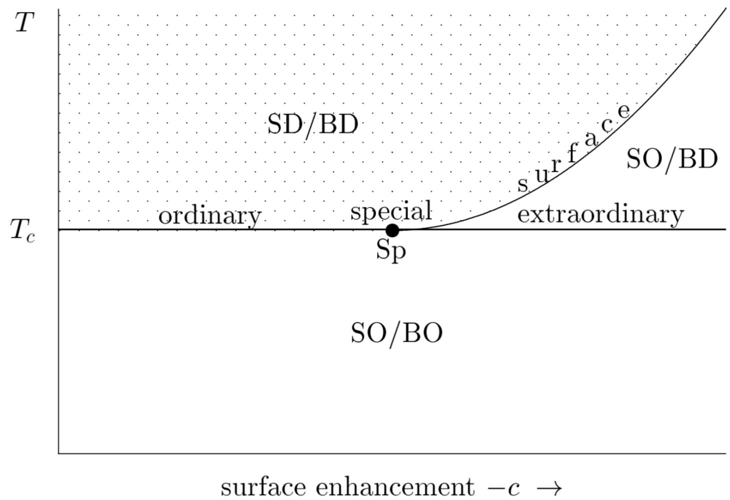

Figure 8: A phase diagram of the critical phenomena associated with the Ising model in 3d with 2d boundary. For $O(N)$, $N > 1$, the surface transition should be a cross-over because of Coleman-Mermin-Wagner. SD = surface disordered, SO = surface ordered, BD = boundary disordered, BO = boundary ordered.

where the surface critical exponent $\hat{\beta}$ is determined by a combination of bulk and boundary data:

$$\hat{\beta} = \frac{\Delta_{\partial_n \hat{\phi}}}{d - \Delta_{\phi^2}} \ . \tag{6.42}$$

The idea here is that $T_c - T$ is the coefficient (or source) of the relevant bulk operator ϕ^2. Moreover, the boundary magnetization should scale with the boundary order parameter, in this case $\partial_n \hat{\phi}$. This ratio $\hat{\beta}$ is then the unique combination of anomalous dimensions (critical exponents) which will produce something of the correct (renormalized) engineering dimension. Without derivation, let us quote results to $O(\epsilon^2)$ for both $\Delta_{\partial_n \hat{\phi}}$ and Δ_{ϕ^2}:[13]

$$\begin{aligned} \Delta_{\phi^2} &= d - 2 + \frac{N+2}{N+8}\epsilon + \frac{13N^2 + 70N + 88}{2(N+8)^3}\epsilon^2 + O(\epsilon^3) \ , && (6.43) \\ \Delta_{\partial_n \hat{\phi}} &= \frac{d}{2} - \frac{1}{2}\frac{N+2}{N+8}\epsilon - \frac{(N+2)(17N+76)}{4(N+8)^3}\epsilon^2 + O(\epsilon^3) \ . && (6.44) \end{aligned}$$

Note we have already discussed and partially derived the $O(\epsilon)$ contributions above. (The bulk quantities are actually known up to $O(\epsilon^5)$, but let us keep things simple here.) From this, we can compute $\hat{\beta}$ for the $N = 0$, 1, 2, and 3 $O(N)$ models. The results are $\hat{\beta} = 0.78$, 0.79, 0.81, and 0.82 respectively, keeping terms to $O(\epsilon^2)$ in $\hat{\beta}$ and boldly setting $\epsilon = 1$ to go to the $d = 3$ limit.

[13] J. S. Reeve and A. J. Guttmann, "Critical Behavior of the n-Vector Model with a Free Surface," Phys. Rev. Lett. **45**, p 1581 (1980).

We can then compare these numbers against experimental measurements. Spin polarized low energy electron diffraction was used to measure the surface magnetization of nickel, which is thought to be in the $O(3)$ universality class. The experimental measurement[14] is $\hat{\beta} = 0.825^{+0.025}_{-0.040}$, which agrees well with our estimate 0.82 above. There is also a grazing incidence x-ray diffraction experiment measuring the surface magnetization of an iron-aluminum alloy,[15] which measures $\hat{\beta} = 0.75 \pm 0.02$. This material is thought to be in the $O(2)$ universality class, for which our estimate of $\hat{\beta} = 0.81$ is not terrible although not great either. Finally, there are experiments with binary liquids and a molecular solid which are supposed to be in the Ising universality class $N = 1$. Here the surface magnetization was measured[16] to be 0.83 ± 0.05 and 0.8 ± 0.1 respectively, in rough agreement with the theoretical prediction 0.79 above.

7 Trace Anomalies

Consider a QFT with classical Weyl symmetry and diffeomorphism invariance. While classical Weyl symmetry supposedly guarantees the stress tensor is traceless, one loop effects will typically introduce a scale dependence to the action and correspondingly a violation of the relation $\langle T^\mu_\mu \rangle = 0$. For example, in QED in 4d one will find $\frac{\beta(e)}{2e} F_{\mu\nu} F^{\mu\nu}$ showing up on the right hand side. Let's assume however that in flat space, we really do have a conformal field theory so that $T^\mu_\mu = 0$. Still, however, in curved space, there may be a violation of this relation. There is a trace anomaly – also known as a Weyl anomaly, scale anomaly, and conformal anomaly. There is a special role for the quantum (or anomalous) dependence of $\langle T^\mu_\mu \rangle$ on the curvature of space-time. Perturbations in the metric source the stress-tensor after all, and the stress-tensor is a universal operator that controls the flow of energy and momentum and is present in all unitary CFTs.

The most famous example of this phenomena is the result for 2d conformal field theories that

$$\langle T^\mu_\mu \rangle = \frac{c}{24\pi} R \ , \tag{7.1}$$

where R is the Ricci scalar curvature and c is the central charge, so called because it also shows up in a central term of the Virasoro symmetry algebra that 2d conformal field theories possess. The importance of this coefficient c for 2d CFT is hard to overstate. Its remarkable properties have motivated extensive investigation of trace anomalies, not only in 2d but in higher dimensional CFT and QFT as well.

Here is a non-exhaustive list of reasons why c is important in 2d CFT:

- There is a c-theorem which states that under renormalization group flow from the UV to the IR, c must decrease.[17] This monotonicity thus places an ordering on the space of 2d QFTs.

[14] S. F. Alvarado, M. Campagna, and H. Hopster, Phys. Rev. Lett. **48**, 51 (1982).

[15] X. Mailänder, H. Dosch, J. Peisl, and R. L. Johnson, Phys. Rev. Lett. **64**, 2527 (1990).

[16] L. Sigl and W. Fenzl, Phys. Rev. Lett. **57**, 2191 (1986) and B. Burandt, W. Press, and S. Haussühl, Phys. Rev. Lett. **71**, 1188 (1993).

[17] A. B. Zamolodchikov, "Irreversibility of the Flux of the Renormalization Group in a 2D Field Theory," JETP Lett. **43**, 730-732 (1986)

- All correlation function of the stress tensor in 2d CFT (in flat space) are fixed once c is known.[18]
- The one-point function $\langle T_{\mu\nu}\rangle$ of a 2d CFT on a general curved manifold is also fixed by c via something called the Schwarzian derivative.
- A corollary of the previous example, for the special case of a cylinder, c determines the equation of state of a 2d CFT (in flat space). The energy density as a function of temperature is $\epsilon = \frac{\pi c T^2}{6}$. (Equivalently, the Casimir energy of a CFT on a circle of circumference L is $E = -\frac{\pi c}{6L}$.)
- The central charge c also controls the entanglement entropy of a single interval of length L, $S_E = \frac{c}{3}\log\frac{L}{\epsilon}$, where ϵ is a short distance UV cut-off.[19]

How then can the trace of the stress tensor (in a diffeomorphism invariant and classically Weyl symmetric theory) depend on curvature? The trace of the stress tensor has units of energy density – in other words, it is an object of scaling dimension d. The trace is furthermore a scalar quantity. A corollary of Weyl invariance, there can be no dimensionful parameters in the definition of the theory, e.g. a mass. Thus, because of diffeomorphism invariance, the only thing we have to work with are polynomials constructed from contractions of and derivatives acting on the Riemann curvature tensor. As the Riemann tensor has scaling dimension 2, we immediately come to the remarkable conclusion that the trace must vanish for odd dimensional theories. (In a moment, the introduction of a boundary or defect will give us a way around this restriction.) In 2d, from this perspective, the only thing that could possibly appear on the right hand side is the Ricci scalar curvature, providing another demonstration of Murphy's Law that anything that can happen, will happen.

The next most complicated example is four dimensions, in which case we may consider the following four terms, $R_{\mu\nu\lambda\rho}R^{\mu\nu\lambda\rho}$, $R_{\mu\nu}R^{\lambda\rho}$, R^2 and $\Box R$. In this case, it turns out there is only a two parameter family of linear combinations of these two terms that contributes in a physically meaningful way to $\langle T^\mu_\mu\rangle$. These two combinations are the Euler density and the the Weyl curvature squared:

$$\langle T^\mu_\mu\rangle = \frac{1}{(4\pi)^2}\left(-a^{(4d)}E_4 + c^{(4d)}I\right) \ , \tag{7.2}$$

where

$$E_4 \;=\; \epsilon^{\mu_1\mu_2\mu_3\mu_4}\epsilon_{\nu_1\nu_2\nu_3\nu_4}R^{\nu_1\nu_2}{}_{\mu_1\mu_2}R^{\nu_3\nu_4}{}_{\mu_3\mu_4} \ , \quad I = W^{\mu\nu\lambda\rho}W_{\mu\nu\lambda\rho} \ . \tag{7.3}$$

The minus sign in front of $a^{(4d)}$ may look peculiar but it has been inserted so that $a^{(4\mathrm{d})} > 0$ and also so that the a-theorem – a 4d analog of the 2d c-theorem – can be stated in the conventional way $a^{(4\mathrm{d})}_{\mathrm{UV}} > a^{(4\mathrm{d})}_{\mathrm{IR}}$.[20] Similar to the 2d case, $c^{(4d)}$ fixes the two-point function

[18] A. M. Polyakov, "Quantum Geometry of Bosonic Strings," Phys. Lett. B **103**, 207-210 (1981)

[19] C. Holzhey, F. Larsen and F. Wilczek, "Geometric and renormalized entropy in conformal field theory," Nucl. Phys. B **424**, 443-467 (1994) [arXiv:hep-th/9403108 [hep-th]].

[20] Z. Komargodski and A. Schwimmer, "On Renormalization Group Flows in Four Dimensions," JHEP **12**, 099 (2011) [arXiv:1107.3987 [hep-th]].

$\langle T_{\mu\nu}(x)T_{\rho\sigma}(0)\rangle$ while $a^{(4d)}$ and $c^{(4d)}$ partially fix the three point function $\langle T_{\mu\nu}(x)T_{\rho\sigma}(x')T_{\lambda\tau}(0)\rangle$. (A third constant is needed to pin down its form completely.)

What principle or principles are at work to single out these two contributions to the trace-anomaly? Wess-Zumino consistency and independence of counter-terms in the effective action, as we now explain in more detail. From the path integral, we have a quantum effective action W, defined such that $e^{-W} = Z = \int [d\phi] e^{-S[\phi]}$, reverting to an Euclidean perspective for simplicity. We can think of the trace of the stress tensor as coming from a Weyl symmetry variation, $g_{\mu\nu} \to e^{2\sigma} g_{\mu\nu}$, of W:[21]

$$\delta_\sigma W = -\int \mathrm{d}^d x \sqrt{g} T^\mu_\mu \delta\sigma \ . \tag{7.4}$$

Now the set of Weyl variations form an Abelian group, and thus it had better be true that $\delta_{\sigma_1}\delta_{\sigma_2} W = \delta_{\sigma_2}\delta_{\sigma_1} W$. The constraint is known as Wess-Zumino consistency, and has more elaborate realizations for anomalies involving non-abelian symmetry groups. It turns out that this innocent looking condition puts severe constraints on what can appear in the trace anomaly. In our case, the $R^2\delta\sigma_1$ term on its own for example will vary into something proportional to $R\delta\sigma_1 \Box \delta\sigma_2$, which is not WZ consistent. The Weyl curvature squared satisfies WZ consistency trivially because the combination $\sqrt{g} I$ is Weyl invariant. The Euler density on the other hand varies to produce a total derivative, and thus is WZ consistent after integrating over space.

The story doesn't end here. QFT is plagued by infinities, and curved space just makes the story worse. To cure these infinities, we are traditionally allowed to add counter-terms to the action. One diffeomorphism invariant possible counter-term is R^2. However, as we just mentioned, R^2 varies to produce $R \Box \delta\sigma$ which is equivalent via integration by parts to the term $(\Box R)\,\delta\sigma$. Thus, if a $\Box R$ term shows up in the trace anomaly, its coefficient will not be physical because it can be shifted by an R^2 counter-term in the action. Depending on one's regularization scheme, $\Box R$ will in general be present in the trace anomaly. However, by abuse of notation, people often suppress it when they write the trace anomaly for 4d CFTs.

Introducing a boundary or defect brings in a couple of new ingredients. There is now a submanifold and the possibility of having a contribution to the trace anomaly that is localized on the defect. Furthermore, there is a new curvature with scaling dimension one, namely the extrinsic curvature $K^i_{\mu\nu}$ (or equivalently the second fundamental form). The existence of the extrinsic curvature means that both even and odd dimensional defects can support contributions to the trace anomaly.

We are getting ahead of ourselves, however. As the dimension increases, the complexity of the calculations increase while the conceptual issues remain largely invariant. Let us then return to 2d CFTs and explain some of the claims made in the bulleted list above for c. We have in the 2d CFT case that

$$\delta_\sigma W = -\frac{c}{24\pi} \int \mathrm{d}^2 x \sqrt{g} R \delta\sigma \ . \tag{7.5}$$

One is then naturally led to ask what W varies to produce this trace anomaly? We cannot

[21] The definition of the stress tensor conventionally picks up an extra minus sign in Euclidean signature.

get at the whole of W by integration, but we can at least compute the difference

$$\mathcal{W} = W[e^{2\sigma}\delta_{\mu\nu}] - W[\delta_{\mu\nu}] \ . \tag{7.6}$$

As any metric in 2d is related to $\delta_{\mu\nu}$ by Weyl rescaling, this difference is more general than at first appears.

Unfortunately, one cannot express $\mathcal{W}$ locally in terms of the metric, but there is a nice, local expression that involves an extra massless field τ, sometimes called the dilaton. We define τ such that it transforms under Weyl symmetry via $\tau \to \tau + \sigma$. A natural guess for the effective action is then

$$\mathcal{W}_0 = -\frac{c}{24\pi}\int \mathrm{d}^2x\sqrt{g}R\tau \ . \tag{7.7}$$

Unfortunately, R also transforms under Weyl transformation. In general d, we have that

$$R[e^{2\sigma}g_{\mu\nu}] = e^{-2\sigma}\left(R[g_{\mu\nu}] - 2(d-1)\Box\sigma - (d-2)(d-1)(\partial\sigma)^2\right) \ , \tag{7.8}$$

where the third term proportional to $d-2$ will of course vanish in $d=2$. An easy way then to cure this extra $\Box\sigma$ piece that does not transform in the right way is to modify our initial guess by a kinetic term for τ:

$$\mathcal{W} = -\frac{c}{24\pi}\int \mathrm{d}^2x\sqrt{g}\left(R[g_{\mu\nu}]\tau - (\partial\tau)^2\right) \ . \tag{7.9}$$

Above we claimed that expressing the anomaly action purely in terms of the metric would give a nonlocal expression. By "integrating out" τ, we can see now how that comes to pass. The equation of motion for τ is $R = -2\Box\tau$. Thus formally at least we may write

$$\mathcal{W} = -\frac{c}{12\pi}\int \mathrm{d}^2x\sqrt{g}R\frac{1}{\Box}R \ . \tag{7.10}$$

This presentation of the anomaly action is sometimes called the Polyakov action. (A similar procedure in 4d and 6d will yield anomaly effective actions involving $(\partial\tau)^4$ and $(\partial\tau)^6$ respectively, plus many more terms.)

This effective action can be used to determine correlation functions of the stress tensor. A more precise way of writing (7.10) is as a double integral

$$\mathcal{W} = -\frac{c}{12\pi}\int \mathrm{d}^2x'\sqrt{g'}\int \mathrm{d}^2x\sqrt{g}R(x)\frac{1}{4\pi}\log|x-x'|^2 R(x') \ , \tag{7.11}$$

where we have used the fact that $\Box\log|x|^2 = 4\pi\delta^{(2)}(x)$. We can for instance now compute the two-point function of the stress-tensor in flat space. We consider a small perturbation to the metric about flat space, $g_{\mu\nu} = \delta_{\mu\nu} + h_{\mu\nu}$. In complex coordinates, the Ricci scalar $R = 4(\partial_{\bar z}^2 h_{zz} + \partial_z^2 h_{\bar z\bar z})$ to first order in $h_{\mu\nu}$. Expanding out the Polyakov action gives

$$\begin{aligned}\mathcal{W} &\approx -\frac{c}{48\pi^2}\int \mathrm{d}^2z\int \mathrm{d}^2z'(\partial_{\bar z}^2\log|z-z'|^2)h_{zz}(z,\bar z)\partial_{\bar z'}^2 h_{zz}(z',\bar z') \\ &= \frac{c}{8\pi^2}\int \mathrm{d}^2z\int \mathrm{d}^2z'\frac{h_{zz}(z,\bar z)h_{zz}(z',\bar z')}{(\bar z-\bar z')^4}\end{aligned} \tag{7.12}$$

from which we can read the two point function

$$\langle T^{zz}(z,\bar{z})T^{zz}(z',\bar{z}')\rangle = \frac{c}{4\pi^2}\frac{1}{(\bar{z}-\bar{z}')^4} \ . \tag{7.13}$$

(Note the stress tensor in 2d CFT is often defined with an extra factor of 2π, thus removing the $4\pi^2$ in the denominator of this last expression.)

Another interesting use of $\mathcal{W}$ is to write down an expression for $\langle T^{\mu\nu}\rangle$ in a general curved background. If we vary the dilaton action (7.9) with respect to the metric, we find that

$$\langle T_{\mu\nu}\rangle_{e^{2\sigma}\delta_{\mu\nu}} - \langle T_{\mu\nu}\rangle_{\delta_{\mu\nu}} = \frac{c}{12\pi}\left[(\partial_\mu\tau)(\partial_\nu\tau) + D_\mu\partial_\nu\tau - g_{\mu\nu}\left(\frac{1}{2}(\partial\tau)^2 + \Box\tau\right)\right] \ . \tag{7.14}$$

Note a piece proportional to $(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R)\tau$ drops out because Einstein's equations are trivial in 2d. Because of the symmetries of the plane, we expect that for the vacuum state, $\langle T_{\mu\nu}\rangle_{\delta_{\mu\nu}} = 0$ and we henceforth drop it from the equations. On-shell, we saw that $-2\Box\tau = R$. In a conformal frame where $g_{\mu\nu} = e^{2\sigma}\delta_{\mu\nu}$, we see that $R[g_{\mu\nu}] = -2\Box\sigma$. Thus on-shell, we can identify $\sigma = \tau$, and the vacuum stress tensor on the manifold with metric $e^{2\sigma}\delta_{\mu\nu}$ becomes

$$\langle T_{\mu\nu}\rangle_{e^{2\sigma}\delta_{\mu\nu}} = \frac{c}{12\pi}\left[-(\partial_\mu\sigma)(\partial_\nu\sigma) + \partial_\mu\partial_\nu\sigma + \delta_{\mu\nu}\left(\frac{1}{2}(\partial\sigma)^2 - \partial\cdot\partial\sigma\right)\right] \ , \tag{7.15}$$

where now we have written everything in terms of a flat metric $\delta_{\mu\nu}$, explaining the two curious sign changes in the equation above.

If we rephrase this result in terms of a conformal transformation that produces this Weyl scaling factor $e^{2\sigma}$, then we recover the usual formula for the Schwarzian derivative. In particular, we have

$$g_{w\bar{w}} = \left(\frac{\partial z}{\partial w}\right)\left(\frac{\partial\bar{z}}{\partial\bar{w}}\right)g_{z\bar{z}} = e^{2\sigma}g_{z\bar{z}} \ . \tag{7.16}$$

Then we find in complex coordinates

$$\begin{aligned}
\langle T_{ww}(w)\rangle_{e^{2\sigma}\delta_{\mu\nu}} &= \frac{c}{12\pi}[-(\partial_w\sigma)^2 + (\partial_w^2\sigma)] \\
&= \frac{c}{24\pi}\frac{2(\partial_w^3 z)(\partial_w z) - 3(\partial_w^2 z)^2}{2(\partial_w z)^2} \\
&= \frac{c}{24\pi}\{z,w\} \ .
\end{aligned} \tag{7.17}$$

This funny object $\{z,w\}$ is the Schwarzian derivative.

Consider a cylinder parametrized by $w = \sigma^1 + i\sigma^2$ where σ^2 is periodic with period β: $\sigma^2 + \beta \sim \sigma^2$. Eventually, we will be able to interpret σ^2 as a Euclidean time coordinate and $\beta = 1/T$ as the inverse temperature, but for now we can treat β as just some length scale characterizing the cylinder. There is a plane to cylinder transformation given by the exponential map $z = e^{2\pi w/\beta}$. Let us see how the stress tensor behaves with respect to this transformation:

$$T_{ww}(w)_{\text{cyl}} = \frac{c}{24\pi}\{z,w\} \ . \tag{7.18}$$

Plugging in the exponential map yields

$$T_{ww}(w)_{\rm cyl} \;=\; \frac{c}{24\pi}\left(\frac{2\pi}{\beta}\right)^2 \frac{2-3}{2} \ . \tag{7.19}$$

It follows from the Schwarzian derivative then that

$$\langle T_{ww}(w)\rangle_{\rm cyl} = -\frac{c}{48\pi}\left(\frac{2\pi}{\beta}\right)^2 \ . \tag{7.20}$$

Translating back to a rectilinear coordinate system, we obtain

$$T^{22} = -T^{11} = T_{zz}(z) + T_{\bar z\bar z}(\bar z)) = -\frac{c}{24\pi}\left(\frac{2\pi}{\beta}\right)^2 \ . \tag{7.21}$$

We can interpret this result in one of two ways. If we think of σ^1 as the Euclidean time coordinate and the CFT as living on a circle of circumference β, then Wick rotating to Minkowski signature, we obtain a negative Casimir energy density

$$T^{tt} = -T^{11} = -\frac{\pi c}{6\beta^2} \ . \tag{7.22}$$

Alternatively, we can treat σ^2 as a Euclidean time direction, in which case $\beta = 1/T$ is interpreted as an inverse temperature. In this case, Wick rotating back, we get a positive thermal energy density

$$T^{tt} = -T^{22} = \frac{\pi c T^2}{6} \ . \tag{7.23}$$

7.1 2d Surfaces and Boundaries

Let us now broaden our perspective and think about the 2d curved manifold as a surface in a larger dimensional geometry. It could be a boundary of a 3d theory, or a defect in a 4d or larger dimensional space-time. As we mentioned above, in addition to the Riemann tensor, we now have an extrinsic curvature (or equivalently second fundamental form) with which to construct curvature invariants that could contribute to the trace anomaly. Furthermore, normal and tangential directions along the defect are distinguished. It turns out that in general one finds two additional physically meaningful surface localized contributions to the trace anomaly:

$$T^\mu_\mu\big|_\Sigma = \frac{1}{24\pi}\left(a_\Sigma \bar R + b_1 \,\mathrm{tr}\,\hat\Pi^2 + b_2 W^{ab}{}_{ab}\right) \ . \tag{7.24}$$

Here $\bar R$ is the intrinsic Ricci scalar on Σ, $\mathrm{tr}\,\hat\Pi^2$ is a scalar quantity constructed from the trace subtracted second fundamental form, $W_{\mu\nu\lambda\rho}$ is the Weyl tensor, and we are using the indices $a, b, c, \ldots$ to index the directions tangential to the surface. Note we have replaced the central charge c with the coefficient a_Σ. In higher dimensional theories, the coefficients of the Euler density terms are conventionally denoted a, and there is no longer an obvious role for a_Σ as the central charge in a symmetry algebra – it is not clear why the theory should have a full Virasoro algebra as the bulk theory without the defect does not.

Let us take a brief detour to recall some facts about submanifolds. Let ξ^a be coordinates on Σ and $X^\mu(\xi^a)$ be the embedding functions. We have an induced metric $\bar{g}_{ab} = e^\mu_a e^\nu_b g_{\mu\nu}$ where $e^\mu_a = \partial X^\mu / \partial \xi^a$. We can introduce a covariant derivative that acts on tensors with mixed indices, $\nabla_a \omega^\mu_c = \partial_a \omega^\mu_c + \Gamma^\mu_{\nu a}\omega^\nu_b - \bar{\Gamma}^c_{ab}\omega^\mu_c$. Here $\Gamma^\mu_{\lambda\rho}$ and $\bar{\Gamma}^a_{bc}$ are the Christoffel symbols constructed from $g_{\mu\nu}$ and $\bar{g}_{ab}$ respectively. Furthermore, we have $\Gamma^\mu_{\nu a} = e^\rho_a \Gamma^\mu_{\nu\rho}$. The second fundamental form is $\Pi^\mu_{ab} = \nabla_a e^\mu_b$, and its traceless version is $\hat{\Pi}^\mu_{ab} = \Pi^\mu_{ab} - \frac{1}{p}\gamma_{ab}\Pi^\mu$ where $\Pi^\mu = \gamma^{ab}\Pi^\mu_{ab}$. In the codimension one case, the extrinsic curvature is related to the second fundamental form $K_{ab} = -n_\mu \Pi^\mu_{ab}$.

At this point in the lectures, I grow tired of providing detailed calculations and instead shall try to give a you a brief tour of what is known about these defect anomalies, starting with the surface case. As before, the form $T^\mu_\mu\big|_\Sigma$ can be shown using Wess-Zumino consistency and independence of local counter-terms.

- $a_{\Sigma,UV} > a_{\Sigma,IR}$: One can again show monotonicity under RG flow, but it's only under RG flow induced by relevant boundary or defect operators. In fact while a_Σ is insensitive to the values of boundary marginal operators, it is very sensitive to bulk marginal operators, making the quantity much less useful as any kind of RG monotone. The derivative of a_Σ with respect to bulk marginal operators is proportional to the corresponding one-point function coefficients.

- b_1 is proportional to the coefficient of the displacement two-point function. The relation can be motivated by noticing that for slightly curved boundaries, the extrinsic curvature is proportional to the Hessian of the embedding function $K_{ab} \sim \partial_a \partial_b X^n$. Thus, a double derivative of the trace anomaly with respect to X^n gives the anomalous scale dependence of $\langle D^n D^n \rangle$ (a contact term) whose normalization unambiguously fixes the normalization of $\langle D^n D^n \rangle$ itself.

- b_2 is proportional to the coefficient of $\langle T^{\mu\nu} \rangle$. Note $\langle T^{\mu\nu} \rangle$ must vanish in the codimension one case and correspondingly the Weyl curvature vanishes in a three dimensions. The Weyl curvature is proportional to a double derivative of the metric. Thus varying the trace anomaly with respect to the metric give the scale dependence of $\langle T^{\mu\nu} \rangle$, whose normalization uniquely fixes $\langle T^{\mu\nu} \rangle$ itself.

One can wonder if there are relations between these anomaly coefficients. Indeed, in the supersymmetric case, it is believed and proven in some cases that $b_1 \sim b_2$. One can ask if there are bounds. Since $b_1 \sim \langle D^i D^i \rangle$, we know $b_1 > 0$ by reflection positivity. There is also something called the averaged null energy condition (ANEC),

$$\int_{-\infty}^{\infty} \langle T_{\mu\nu} \rangle v^\mu v^\nu du \geq 0 \tag{7.25}$$

where v^μ is a tangent to a null geodesic with affine parameter u. Thus, for theories that obey the ANEC we expect $b_2 > 0$.

The story continues in higher dimensions. For example a 3d boundary with a 4d bulk has a contribution to the trace of the schematic form

$$T^\mu_\mu\big|_{\text{bry}} = b_1 \hat{K} W + b_2 \hat{K}^3 \tag{7.26}$$

where $\hat{K}$ is the trace removed extrinsic curvature and W is again the Weyl curvature. Now it turns out that b_1 is proportional the displacement two point function coefficient (and also to the coefficient of $\langle T^{\mu\nu} D^n \rangle$). b_2 on the other hand is fixed by the coefficient of $\langle D^n D^n D^n \rangle$. In the 4d case, the story gets substantially more complicated. There are order 20 invariants contributing to T^μ_μ, only three of which we understand pretty well. There is a $\nabla \hat{\Pi} W$ term which is fixed by $\langle D^i D^i \rangle$. There is a $\nabla^2 W$ term which is fixed by $\langle T^{\mu\nu} \rangle$. And there is of course an Euler density term $\overline{E}_4$ and associated monotonicity theorem $a_{\Sigma,UV} > a_{\Sigma,IR}$. The paper just came out last week though, and there is a lot of work still to be done.

8 Mixed Dimensional QED

[[A very interesting story, but we have run out of time. . .]]

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A Sources

I set the level and course material largely using this set of notes:

- L. F. Alday, "Conformal Field Theory", class notes from a set of lectures delivered at Oxford University,

  courses.maths.ox.ac.uk/node/view_material/5310

Here are further references on conformal field theory:

- P. Di Francesco, P. Mathieu, and D. Sénéchal, *Conformal Field Theory*, Springer, 1997.

  The canonical reference for conformal field theory, also called "the yellow book". The early chapters cover CFT in general dimension and are useful for this course. The later chapters, which constitute most of the book, are devoted to CFT in $d = 2$.

- P. Ginsparg, "Applied Conformal Field Theory,"

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- S. Rychkov, "EPFL Lectures on Conformal Field Theory in $D \geq 3$ Dimensions,"

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Covers roughly the same material that is in this course, but targeted toward more advanced graduate students.

- J. Cardy, “Conformal Field Theory and Statistical Mechanics,” Les Houches Lecture Notes, 2008

  www-thphys.physics.ox.ac.uk/people/JohnCardy/

  In fact a variety of lecture notes are available from the home page of this master of conformal field theory.